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# Forced nonlinear oscillations of semi-infinite cables and beams resting on a unilateral elastic substrate

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**Abstract**

In this work, we study the nonlinear oscillations of mechanical systems resting on a (unilateral) elastic substrate reacting in compression only. We consider both semi-infinite cables and semi-infinite beams, subject to constant distributed load and to a harmonic displacement applied to the finite boundary. Due to the nonlinearity of the substrate, the problem falls in the realm of free-boundary problems, because the position of the points where the system detaches from the substrate, called Touch Down Points (TDP), is not known in advance. By an appropriate change of variables, the problem is transformed into a fixed-boundary problem, which is successively approached by a perturbative expansion method. In order to detect the main mechanical phenomenon, terms up to the second order have to be considered. Two different regimes have been identified in the behaviour of the system, one below (called subcritical) and one above (called supercritical) a certain critical excitation frequency. In the latter, energy is lost by radiation at infinity, while in the former this phenomenon does not occur and various resonances are observed instead; their number depends on the statical configuration around which the system performs nonlinear oscillations.

## 1 Introduction

This work is aimed at studying the nonlinear forced oscillations of mechanical systems resting on a (unilateral) elastic substrate reacting in compression only. We consider both semi-infinite cables and semi-infinite beams, subject to a constant distributed load and to a harmonic displacement applied to the finite boundary.

Our original motivation was that of describing the laying of marine pipelines [2]. ¿From an engineering point of view, the study of the mechanical behaviour of pipelines during the laying phase is crucial to avoid failures and damages, this phase being the most demanding in terms of mechanical strength. Some models were proposed by Lenci and Callegari in [7], and analytical solutions were found in the static case. Here, we study the dynamic behaviour of pipelines which undergo vertical motions due, e.g., to the oscillations of the barge on the sea surface.

Because this problem is too difficult to be handled with analytical techniques, we focus on two prototype problems, which, on the one hand, are governed by easier, piece-wise linear equations, and, on the other hand, are able to describe some of the mechanical phenomena of the original problem we wish to investigate.

More specifically, we focus our attention on the laid part of the pipe and on the first part of the suspended span. These two portions are divided by the so-called Touch-Down Point (TDP) (Fig. 1). We refer to [4] for the study of the motion of the suspended part from the TDP to the laying barge. Our simplifications are motivated by the fact that these models capture two main sources of difficulties, namely, the semi-infinite length of the laid beam and the nonlinearity due to the unilateral behaviour of the springs, whose combined study is the subject of the present work.

The problem considered here falls in the realm of unilateral problems [1], also known as "moving" or "free" boundary problems [5]; it arises in various engineering applications, e.g., in the field of the dynamics of soil-foundation interactions [16] and in the dynamics of railways tracks, and it has an interest *per se*, because it is able to detect in a simple way complex dynamical behaviours. This explains the reason for a joint study of cables and beams, which is that of investigating how the considered phenomena depend on the specific mechanical model.

Analytical and numerical solutions for the dynamics of a beam on unilateral elastic springs can be found, e.g., in [15] and [3], respectively. A more mathematically oriented approach can be found in [13]. The dynamics is governed by a moving-boundary problem where the position of the TDP is an additional unknown. Since there is no hope to find an exact solution, because of the nonlinearity, we look for an approximate solutions by using asymptotic analysis [9, 10]. Perturbation techniques were previously applied to study the nonlinear dynamics of finite length beams (see, e.g., [11], where the method of multiple scales is used to attack directly the integro-partial differential equation of motion). In this work the extension to infinite length is considered.

In our perturbation expansion, the zero order terms correspond to the static solution obtained in the absence of a time-dependent excitation applied at the boundary, and are the starting point of the analysis. The first order terms are the most important ones, and permit to understand the resonance behaviour of the system and the questions related to the wave propagation toward infinity. In particular, these terms permit to identify two different regimes, below and above a certain critical excitation frequency, with very different wave properties [6].

The second order terms give information on the nonlinear coupling between various modes. Their computation is very hard, and this task will be left for a future work. However, much information on their behaviour can be inferred from the analysis of their governing equations; these aspects have been verified numerically elsewhere [8] for the case of the beam equation.

## 2 The mathematical models

Here, we introduce the differential equations that govern the time-dependent behaviour of the profiles of the cables (wave equation) or beams (beam equation). In either case, the profile is represented by the function $u(x,t)$, where $0 \leq x < +\infty$ is the space variable and $t \geq 0$ the time. A restoring force, with elastic constant $k$, acts only on the portion of the spatial domain where the solution $u(x,t)$ is negative. This describes the action of the elastic substrate that acts in compression only. In our models, we assume that there exists only one point of the domain, $x = c$, where the profile function vanishes, namely $u(c,t) = 0$; moreover, the boundary conditions are such that $u(x,t) > 0$ for $0 \le x < c$ and $u(x,t) < 0$ for $c < x < \infty$. The mechanical systems is shown schematically in Fig. 1.

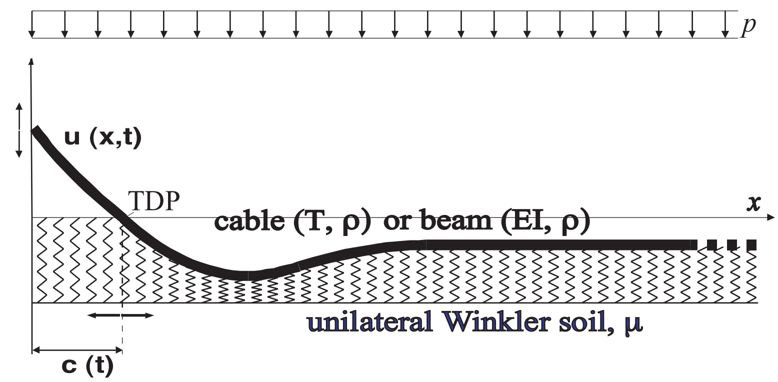


Figure 1: A schematic picture of the considered mechanical systems.

In our analysis, and for both models, the static solution $u(x,t) \equiv u_S(x)$ plays an important role. In this case, $c = c_0$ is a constant, while for the time-dependent solutions we have that $c = c(t)$ is a function of time. The point $x = c$ is called Touch-Down-Point (TDP) in the applications, and we adopt this terminology as well. Suitable continuity conditions on the function $u$ and its spatial derivatives at $x = c$ are also imposed. A constant load $p$, representing the compound action of the gravity acceleration and of the hydrostatic push, is also added to the equations.

In this work, we shall look for time-dependent solutions of the boundary-value problem that correspond to small oscillations about the static solution. These oscillations are induced by a time-dependent boundary condition at $x = 0$, which we will assume of harmonic behaviour. The TDP $x = c(t)$ will then exhibit oscillating behaviour as well, and the main quantity of interest in our work is the ratio of the amplitues of the oscillation of the TDP and the oscillation of the boundary. We shall define these quantities in Section 4.

## 2.1 Wave equation

If the mechanical system is given by taut inextensible cables, the governing equation is the wave equation with the addition of suitable terms describing the constant load and the restoring force,

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + p = 0, \qquad x < c(t) \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + \gamma^2 u + p = 0, \qquad x > c(t), \tag{2}$$

where $v = \sqrt{T/\rho}$ is the propagation speed ($T$ is the axial force and $\rho$ the mass per unit length), $\gamma = \sqrt{k/\rho}$ ($k$ is the mechanical stiffness of the unilateral springs) and $p = \widetilde{p}/\rho$ ($\widetilde{p}$ is the transversal, uniformly distributed, static load). Note that the same equations govern the problem of axial vibrations of rods [12] (in which case $v = \sqrt{EA/\rho}$, $EA$ being the axial stiffness of the rod), and that equation (2) is also known as Klein-Gordon equation.

The boundary condition at $x = 0$ is $u(0,t) = \tilde{U}_0(1 + \varepsilon \sin \Omega t)$, while at $x \to \infty$ we require that $u(x,t)$ be bounded; moreover, we assume that, whenever the equations support travelling-wave solutions, terms corresponding to waves returning from $+\infty$ are not present, so that only "outgoing" waves (travelling to the right) are admitted. Finally, the additional continuity conditions at $x = c$ are

$$u(c^-, t) = u(c^+, t) = 0 \tag{3}$$

$$\frac{\partial u}{\partial x}(c^-, t) = \frac{\partial u}{\partial x}(c^+, t), \tag{4}$$

where $c = c_0$ for static solutions and $c = c(t)$ for time-dependent solutions. Also, with $c^-$ and $c^+$ we indicate the limits of $x \to c$ from the left and from the right, respectively.

It is convenient to write the equations and the boundary conditions in dimensionless form. It is easily seen that, if we introduce the dimensionless variables $\hat{t}$, $\hat{x}$ and $\hat{u}$, given by $\hat{t} = \gamma t$, $\hat{x} = x\gamma/(v\sqrt{2})$ and $\hat{u} = (\gamma^2/p)u$, equations (1) and (2) can be cast in the "universal" form

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + 1 = 0, \qquad x < c(t) \tag{5}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u + 1 = 0, \qquad x > c(t), \tag{6}$$

where we have omitted the hat in order not to burden the notation. The boundary condition at $x = 0$ now becomes

$$u(0,t) = U_0(1 + \varepsilon \sin \omega t), \tag{7}$$

where $U_0 = \gamma^2 \tilde{U}_0/p$ and $\omega = \Omega/\gamma$. Therefore, we see that the problem depends only upon two dimensionless parameters, $\omega$ and $U_0$, entirely included in the boundary condition at

$x = 0$, while the model equations are free of parameters. The continuity conditions are still given by equations (3)-(4).

### 2.2 Beam equation

If the mechanical system is given by beams, the governing equation is the beam equation with the addition of suitable terms describing gravity and the restoring force; in the original physical variables, the equations are

$$\frac{\partial^2 u}{\partial t^2} + b^2 \frac{\partial^4 u}{\partial x^4} + p = 0, \qquad x < c(t) \tag{8}$$

$$\frac{\partial^2 u}{\partial t^2} + b^2 \frac{\partial^4 u}{\partial x^4} + \gamma^2 u + p = 0, \qquad x > c(t), \tag{9}$$

where $b^2 = \sqrt{EI/\rho}$ is a constant ($EI$ is the bending stiffness of the beam), while $\gamma$ and $p$ have the same meaning as in the wave equation. The boundary conditions are

$$u(0,t) = \tilde{U}_0(1 + \varepsilon \sin \Omega t) \tag{10}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0 \tag{11}$$

$$u(x,t) \qquad \text{bounded as } x \to \infty \tag{12}$$

and the additional continuity conditions at $x = c$ are in this case

$$u(c^-,t) = u(c^+,t) = 0 \tag{13}$$

$$\frac{\partial u}{\partial x}(c^-,t) = \frac{\partial u}{\partial x}(c^+,t), \tag{14}$$

$$\frac{\partial^2 u}{\partial x^2}(c^-,t) = \frac{\partial^2 u}{\partial x^2}(c^+,t), \tag{15}$$

$$\frac{\partial^3 u}{\partial x^3}(c^-,t) = \frac{\partial u^3}{\partial x^3}(c^+,t). \tag{16}$$

Again, we assume that the solution is bounded at infinity and that there are no travelling waves returning from $+\infty$. In order to cast the equations in dimensionless form, we again introduce the dimensionless variables $\hat{t}$, $\hat{x}$ and $\hat{u}$ given by $\hat{t} = \gamma t$, $\hat{x} = x\sqrt{\gamma/2b}$ and $\hat{u} = (\gamma^2/p)u$. Equations (8) and (9) then become (the hat has again been omitted to simplify the notation)

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{4}\frac{\partial^4 u}{\partial x^4} + 1 = 0, \qquad x < c(t) \tag{17}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{4}\frac{\partial^4 u}{\partial x^4} + u + 1 = 0, \qquad x > c(t), \tag{18}$$

which are again free of parameters. The boundary condition at $x = 0$ takes the form (7) as for the wave equation, and the problem is seen again to depend upon the same two parameters $\omega$ and $U_0$, with $U_0 = \gamma^2 \tilde{U}_0/p$ and $\omega = \Omega/\gamma$ again. The continuity conditions are still given by equations (13)-(16).

# 3 The static solutions

The static solutions of the model equations play a very important role in our analysis, since they give the zero-order terms in our perturbative approach. In this Section, we derive these solutions for the wave equation and for the beam equation.

## 3.1 Wave equation

We begin by considering equations (5)-(6), in which we switch off the time derivatives, thus obtaining the static equations

$$-\frac{1}{2}u_S'' + 1 = 0, \qquad x < c_0 \tag{19}$$

$$-\frac{1}{2}u_S'' + u_S + 1 = 0, \qquad x > c_0, \tag{20}$$

where we have indicated with $u_S(x)$ the static solution and by $c_0$ the TDP, which is fixed in this case. The boundary condition at $x = 0$, in this stationary case, is now $u_S(0) = U_0$, while the continuity conditions imply $u_S(c_0^-) = u_S(c_0^+) = 0$ and $u_S'(c_0^-) = u_S'(c_0^+)$. Equations (19) and (20) with the assigned boundary and continuity conditions are easily integrated, giving

$$u_S(x) = (x - c_0)\left(x - \frac{U_0}{c_0}\right), \qquad x < c_0 \tag{21}$$

$$u_S(x) = e^{(c_0 - x)\sqrt{2}} - 1, \qquad x > c_0 \tag{22}$$

with the TDP $c_0$ given by

$$c_0 = \frac{\sqrt{2}}{2}\left(\sqrt{1 + 2U_0} - 1\right), \tag{23}$$

which depends only upon $U_0$.

### 3.2 Beam equation

For the beam equations (17)-(18), by switching off the time derivatives we obtain:

$$\frac{1}{4}u_S^{(IV)} + 1 = 0, \qquad x < c_0 \tag{24}$$

$$\frac{1}{4}u_S^{(IV)} + u_S + 1 = 0, \qquad x > c_0. \tag{25}$$

The boundary conditions are the same as for the stationary wave equation with the additional condition $u_S''(0) = 0$, while for the continuity conditions we now add $u_S''(c_0^-) = u_S''(c_0^+)$ and $u_S'''(c_0^-) = u_S'''(c_0^+)$. Equations (24) and (25) with the assigned boundary and continuity conditions are easily integrated, giving

$$u_S(x) = (c_0 - x)\left[\frac{x^3 - (2 + c_0)(c_0 + x)x}{6} + \frac{U_0}{c_0}\right], \qquad x < c_0 \tag{26}$$

$$u_S(x) = e^{(c_0 - x)}\left[\cos(x - c_0) - c_0 \sin(x - c_0)\right] - 1, \qquad x > c_0 \tag{27}$$

with $c_0$ given by the solution of the quartic equation

$$c_0^4 + 4c_0^3 + 6c_0^2 + 6c_0 - 6U_0 = 0, \tag{28}$$

which again depends only upon $U_0$. Equation (28) can also be thought of as a linear equation for $U_0$, with $c_0$ playing the role of a free parameter.

In Figure 2(A) we show the TDP as a function of $U_0$ for the wave equation (solid line) and for the beam equation (dashed line), for $0 \le U_0 \le 100$ and in Figure 2(B) we show the static solution $u_S(x)$, $0 \le x \le 10$, for the wave equation (solid line) and for the beam equation (dashed line) for $U_0 = 10$. Note that the TDP for the wave equation lies to the right of the TDP for the beam equation for all the values of $U_0$ considered here.

## 4 Perturbative approach to the time-dependent model

The moving boundary conditions at $x = c(t)$ for equations (5)-(6) (wave equation), or equations (17)-(18) (beam equation) make the problem very hard to approach. However, since we are interested in motions corresponding to small deviations from the static solution, we approach the problem by perturbative expansions. Before applying the perturbative expansion, however, we perform a variable transformation which maps the

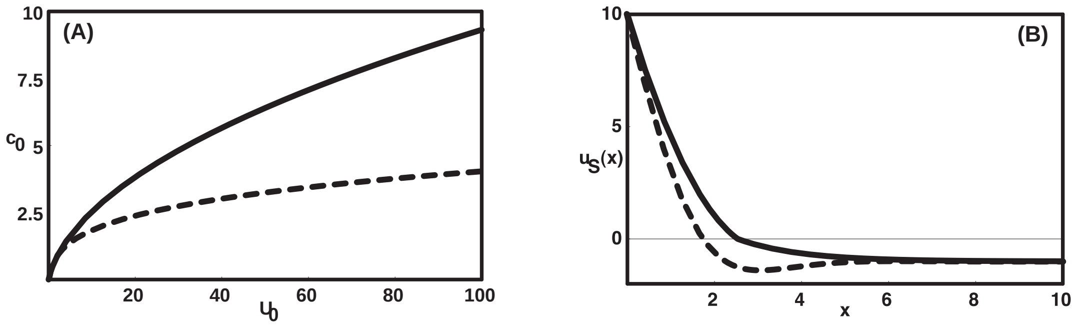


Figure 2: (A) The Touch-Down-Point $c_0$ for the wave equation (solid line) and for the beam equation (dashed line) as a function of $U_0$ for $0 \leq U_0 \leq 100$ and (B) the static solution $u_S(x)$ for the wave equation (solid line) and for the beam equation (dashed line) as a function of $x$, $0 \leq x \leq 10$, for $U_0 = 10$.

original moving-boundary problem into a fixed-boundary problem, which is then amenable to asymptotic analysis.

We seek a transformation from the original variables $(x, t)$ to a new set of variables, $(z, \tau)$, which maps the line $x = c(t)$ of the $(x, t)$ plane into the line $z =$*constant* in the new $(z, \tau)$ plane. Of course, there are many transformations that possess this property; we choose the simplest of all, namely

$$z = \frac{x}{c(t)} \tag{29}$$

$$\tau = t \tag{30}$$

$$u(x, t) = u(zc(t), t) = U(z, t), \tag{31}$$

in which we keep the original time variable. The same transformation has been used, in a different context, by Yilmaz [14]. In order not to burden the notation and not to dirty the pages, we shall not change the notation for the function $u$, which will now be denoted as $u(z, t)$, instead of $U(z, t)$. With this transformation, the moving TDP $x = c(t)$ becomes $z = 1$ and is now fixed, so that $u(1, t) = 0$, while $x = 0$ and $x \to \infty$ correspond to $z = 0$ and $z \to \infty$.

After performing the variable transformation to the wave equation and to the beam equation, we expand the unknown function $u(z, t)$ and the location of the TDP $c(t)$ (unknown as well) in powers of $\varepsilon$ according to

$$u(z, t) = u_0(z) + \varepsilon u_1(z, t) + \varepsilon^2 u_2(z, t) + \varepsilon^3 u_3(z, t) + \mathcal{O}(\varepsilon^4) \tag{32}$$

$$c(t) = c_0 + \varepsilon c_1(t) + \varepsilon^2 c_2(t) + \varepsilon^3 c_3(t) + \mathcal{O}(\varepsilon^4). \tag{33}$$

Note that the zero-order terms are independent of time and therefore, with this expansion, the zero-order quantities will be given by the solutions of the static problems, consistently with our search for solutions near the static profiles.

In general, we shall look for solutions that correspond to small periodic oscillations about the static solutions. Therefore, we shall assume that the functions $u_k(z,t)$ and $c_k(t)$, $k \geq 1$, admit a Fourier-like expansion of the type

$$u_k(z,t) = g_{k0}(z) + \sum_{n=1}^{\infty} \left[ f_{kn}(z) \sin n\omega t + g_{kn}(z) \cos n\omega t \right] \tag{34}$$

$$c_k(t) = b_{k0} + \sum_{n=1}^{\infty} \left[ a_{kn} \sin n\omega t + b_{kn} \cos n\omega t \right]. \tag{35}$$

Our aim is to determine the coefficients $f$, $g$, $a$ and $b$ of these expansions.

As we mentioned in the Introduction, the main quantity of interest for us is the ratio of the amplitude of the oscillation of the TDP and the oscillation of the boundary at $z = 0$. If we denote by $\Gamma(\varepsilon)$ the maximum elongation of the TDP from the static position $c_0$, we introduce the function

$$D(U_0, \omega) = \frac{\Gamma(\varepsilon)}{\varepsilon U_0}; \tag{36}$$

$D$ is called *amplification factor*, since it gives the measure of the amplification with which the oscillation at the boundary is reflected on the oscillation of the TDP. In general, the amplitude $\Gamma$ inherits the expansion in powers of $\varepsilon$ from (33), starting from a first-order term, and therefore the amplification factor also admits a similar expansion, starting from a term of order zero in $\varepsilon$.

### 4.1 Wave equation

We begin by performing the variable transformation (29)-(31) on the wave equation (5)-(6). After evaluating all composed derivatives by using the chain rule, we obtain the transformed differential equations for the new unknown function $u(z,t)$:

$$c^2 \frac{\partial^2 u}{\partial t^2} + \left( \dot{c}^2 z^2 - \frac{1}{2} \right) \frac{\partial^2 u}{\partial z^2} - 2c\dot{c}z \frac{\partial^2 u}{\partial t \partial z} + \left( 2\dot{c}^2 - c\ddot{c} \right) z \frac{\partial u}{\partial z} + c^2 = 0, \qquad z < 1 \tag{37}$$

$$c^2\frac{\partial^2 u}{\partial t^2}+\left(\dot{c}^2 z^2-\frac{1}{2}\right)\frac{\partial^2 u}{\partial z^2}-2c\dot{c}z\frac{\partial^2 u}{\partial t\partial z}+$$
$$\left(2\dot{c}^2-c\ddot{c}\right)z+\frac{\partial u}{\partial z}+c^2(1+u)=0, \qquad z>1 \tag{38}$$

with the boundary conditions

$$u(0,t)=U_0(1+\varepsilon\sin\omega t) \tag{39}$$
$$u(z,t) \qquad \text{bounded as } z\to\infty \tag{40}$$

and the additional continuity conditions

$$u(1^-,t)=u(1^+,t)=0 \tag{41}$$
$$\frac{\partial u}{\partial z}(1^-,t)=\frac{\partial u}{\partial z}(1^+,t). \tag{42}$$

Next, we introduce the perturbative expansion (32)-(33) into the transformed equations (37)-(38) and obtain the usual hierarchy of equations by equating to zero the coefficients of the powers of $\varepsilon$. The details of the calculations are very long and tedious, and we have carried them out with the help of a symbolic manipulation program.

To order $\varepsilon^0$ we have:

$$1-\frac{{u_0}''(z)}{2\,{c_0}^2}=0, \qquad z<1 \tag{43}$$
$$1+u_0(z)-\frac{{u_0}''(z)}{2\,{c_0}^2}=0; \qquad z>1 \tag{44}$$

to order $\varepsilon^1$:

$$\frac{\partial^2 u_1}{\partial t^2}-\frac{1}{2c_0^2}\frac{\partial^2 u_1}{\partial z^2}-\frac{z\,{u_0}'(z)\,\ddot{c}_1(t)}{c_0}+\frac{c_1(t)\,{u_0}''(z)}{{c_0}^3}=0, \qquad z<1 \tag{45}$$
$$\frac{\partial^2 u_1}{\partial t^2}-\frac{1}{2c_0^2}\frac{\partial^2 u_1}{\partial z^2}+u_1(z,t)-\frac{z\,{u_0}'(z)\,\ddot{c}_1(t)}{c_0}+\frac{c_1(t)\,{u_0}''(z)}{{c_0}^3}=0; \qquad z>1 \tag{46}$$

to order $\varepsilon^2$:

$$\frac{\partial^2 u_2}{\partial t^2}-\frac{1}{2c_0^2}\frac{\partial^2 u_2}{\partial z^2}-\frac{z\,{u_0}'(z)\,\ddot{c}_2(t)}{c_0}+\frac{c_2(t)}{c_0^3}u_0''(z)-\frac{3c_1(t)^2}{2c_0^4}u_0''(z)+$$
$$\frac{2\,z\,\dot{c}_1(t)^2\,{u_0}'(z)}{{c_0}^2}+\frac{z\,c_1(t)\,{u_0}'(z)\,\ddot{c}_1(t)}{{c_0}^2}+\frac{z^2\,\dot{c}_1(t)^2\,{u_0}''(z)}{{c_0}^2}-$$
$$\frac{z\,\ddot{c}_1(t)}{c_0}\frac{\partial u_1}{\partial z}-\frac{2\,z\,\dot{c}_1(t)}{c_0}\frac{\partial^2 u_1}{\partial z\partial t}+\frac{c_1(t)}{{c_0}^3}\frac{\partial^2 u_1}{\partial z^2}=0, \qquad z<1 \tag{47}$$

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{1}{2c_0^2}\frac{\partial^2 u_2}{\partial z^2} + u_2(z,t) - \frac{z\, u_0{}'(z)\, \ddot{c}_2(t)}{c_0} + \frac{c_2(t)}{c_0^3} u_0''(z) - \frac{3c_1(t)^2}{2c_0^4} u_0''(z) +$$
$$\frac{2\, z\, \dot{c}_1(t)^2\, u_0{}'(z)}{{c_0}^2} + \frac{z\, c_1(t)\, u_0{}'(z)\, \ddot{c}_1(t)}{{c_0}^2} + \frac{z^2\, \dot{c}_1(t)^2\, u_0{}''(z)}{{c_0}^2} -$$
$$\frac{z\, \ddot{c}_1(t)}{c_0}\frac{\partial u_1}{\partial z} - \frac{2\, z\, \dot{c}_1(t)}{c_0}\frac{\partial^2 u_1}{\partial z \partial t} + \frac{c_1(t)}{{c_0}^3}\frac{\partial^2 u_1}{\partial z^2} = 0. \qquad z > 1 \qquad (48)$$

The boundary conditions associated with this hiererchy of equations are

$$\begin{aligned} u_0(0) &= U_0 & u_0(1) &= 0 \\ u_1(0,t) &= U_0 \sin \omega t & u_1(1,t) &= 0 \\ u_2(0,t) &= 0 & u_2(1,t) &= 0, \end{aligned} \qquad (49)$$

while the continuity conditions on the derivatives (42) have to hold at all orders.

#### 4.1.1 Zero-order solution

Equations (43)-(44), with the boundary conditions for $u_0$ given in (49) are easily integrated giving

$$u_0(z) = c_0 \left( c_0 z - \frac{U_0}{c_0} \right) (z-1)\,, \qquad z < 1, \qquad (50)$$
$$u_0(z) = e^{c_0(1-z)\sqrt{2}} - 1, \qquad z > 1. \qquad (51)$$

It is easy to see that these two equations define the same function given in (21)-(22) as the static solution of the problem, provided that the identification $x = c_0 z$ between the new and the old variables is made. The value of $c_0$ is then obtained by using the continuity condition on the derivative, eq. (42), and is consistent with equation (23).

#### 4.1.2 First-order solution

We obtain the first order solution by substituting the expansions (34)-(35) for $u_1$ and $c_1$ in equations (45)-(46), and then equating separately to zero the coefficient of $\cos n\omega t$ and $\sin n\omega t$ for each $n$. In this way, we obtain an infinite set of equations from which the expansion coefficients $f_{1n}(z), g_{1n}(z), a_{1n}$ and $b_{1n}$ are determined. Again, the calculations are rather long and we carried them out with a symbolic manipulation program. The functions $f_{1n}(z)$ and $g_{1n}(z)$ satisfy non-homogeneous second-order differential equations, in which the known term is proportional to some of the coefficients $a_{1n}$ and $b_{1n}$, and are equipped with the boundary conditions

$$\begin{aligned}
&f_{11}(0) = U_0 \\
&f_{1n}(0) = 0, \qquad && n \neq 1 \\
&g_{1n}(0) = 0, && \forall n \\
&f_{1n}(1) = g_{1n}(1) = 0 && \forall n.
\end{aligned}$$

In addition, the requirements that the functions be bounded as $z \to \infty$ and that there are no waves travelling to the left have to be imposed. We show here the differential equation governing $f_{11}(z)$, which gives the only non-vanishing contribution to $u_1(z,t)$:

$$f_{11}'' + 2c_0^2\omega^2 f_{11} + \left[z\omega^2 c_0 U_0 - 2c_0 + z\omega^2(1-z)c_0\right] a_{11} = 0, \qquad z < 1 \tag{52}$$

$$f_{11}'' + 2c_0^2(\omega^2 - 1) f_{11} + \left(z\omega^2 c_0^2\sqrt{2} - 4c_0\right) e^{c_0(1-z)/\sqrt{2}}\, a_{11} = 0, \quad z > 1. \tag{53}$$

We notice that the solutions of equation (53) depend crucially upon $\omega$: the two linearly independent solutions of the associated homogeneous equation are of hyperbolic type ("subcritical" case) if $\omega < 1$ and of oscillatory type ("supercritical" case) if $\omega > 1$. As shown by the analysis of the higher order terms, the threshold between subcritical and supercritical behaviour is different at different orders in $\varepsilon$; for the second-order terms, for example, the transition occurs at $\omega = 1/2$. In the subcritical case, the boundary conditions that the solution be bounded as $z \to \infty$ must be used, while in the supercritical case we must ensure that there are no travelling waves returning from infinity.

**Subcritical case** ($\omega < 1$).
Due to the fact that $f_{11}$ is the only function which satisfies non-homogeneous boundary conditions for $z < 1$ and by matching the left and right derivatives at $z = 1$, we find that $f_{11}$ and $a_{11}$ are the only non-vanishing contributions to the solution for $u_1(z,t)$ and $c_1(t)$; therefore we obtain

$$u_1(z,t) = f_{11}(z)\sin\omega t \tag{54}$$

$$c_1(t) = a_{11}\sin\omega t, \tag{55}$$

where $f_{11}(z)$ is the solution of equations (52)-(53) with the assigned boundary conditions; the coefficient $a_{11}$ is then determined by the continuity condition on $f_{11}'(z)$ at $z = 1$. We have:

$$a_{11} = \frac{A_-(U_0,\omega)}{B_-(U_0,\omega)} \tag{56}$$

where

$$A_{-}(U_0,\omega) = U_0\,\omega^5\,\csc(\sqrt{2}\,\omega\,c_0)\,c_0^2$$

$$B_{-}(U_0,\omega) = \frac{(U_0 - c_0^2)\,\omega^4}{\sqrt{2}} + \omega^5\,c_0^3\cot(\sqrt{2}\,\omega\,c_0) - \omega\,c_0\,\left(U_0\,\omega^4\,\cot(\sqrt{2}\,\omega\,c_0) + \omega^3\,\left(1 + c_0\sqrt{2(1-\omega^2)}\right)\right)$$

**Supercritical case** ($\omega > 1$).

Also in this case, only terms with $n = 1$ are present in the solution, like in the subcritical case. This time, however, by taking the boundary conditions into account and by matching the left and right derivatives of $f_{11}$ and $g_{11}$ at $z = 1$, we find that the terms proportional to $\cos\omega t$ are also present, and the solution is given by

$$u_1(z,t) = f_{11}(z)\sin\omega t + g_{11}(z)\cos\omega t \tag{57}$$

$$c_1(t) = a_{11}\sin\omega t + b_{11}\cos\omega t, \tag{58}$$

with $a_{11}$ and $b_{11}$ given by

$$a_{11} = \frac{A_{+}(U_0,\omega)}{B_{+}(U_0,\omega)} \tag{59}$$

$$b_{11} = \frac{C(U_0,\omega)}{B_{+}(U_0,\omega)} \tag{60}$$

where

$$A_{+}(U_0,\omega) = U_0\omega c_0^3\left[1 + U_0\omega\cot(\sqrt{2}\omega c_0) - \frac{U_0 - c_0^2}{c_0\sqrt{2}} - \omega c_0^2\cot(\sqrt{2}\omega c_0)\right]$$

$$B_{+}(U_0,\omega) = 2(\omega^2 - 1)\,c_0^4 + \left[\frac{U_0 - c_0^2}{\sqrt{2}} - \left(1 + U_0\omega\cot(\sqrt{2}\omega c_0)\right) + \omega c_0^3\cot(\sqrt{2}\omega c_0)\right]^2$$

$$C(U_0,\omega) = \sqrt{2}U_0\,c_0^4\,\omega\sqrt{\omega^2-1}\,\csc(\sqrt{2}\,\omega\,c_0)$$

To first order in $\varepsilon$, the maximum elongation $\Gamma(\varepsilon)$ of the TDP is given by $\Gamma = |a_{11}|\,\varepsilon$ in the subcritical case, and by $\Gamma = \sqrt{a_{11}^2 + b_{11}^2}\,\varepsilon$ in the supercritical case. In Figure 3, we show the amplification factor $D(U_0,\omega)$ as a function of $\omega$ for three different values of $c_0$: $c_0 = 1.02$ (solid line, corresponding to $U_0 = 2.4$), $c_0 = 2$ (dashed line, corresponding to $U_0 = 6.8$) and $c_0 = 6$ (dotted line, corresponding to $U_0 = 44.5$). The results show the presence of resonances, all lying in the subcritical region, and whose number increases with increasing $U_0$ or $c_0$.

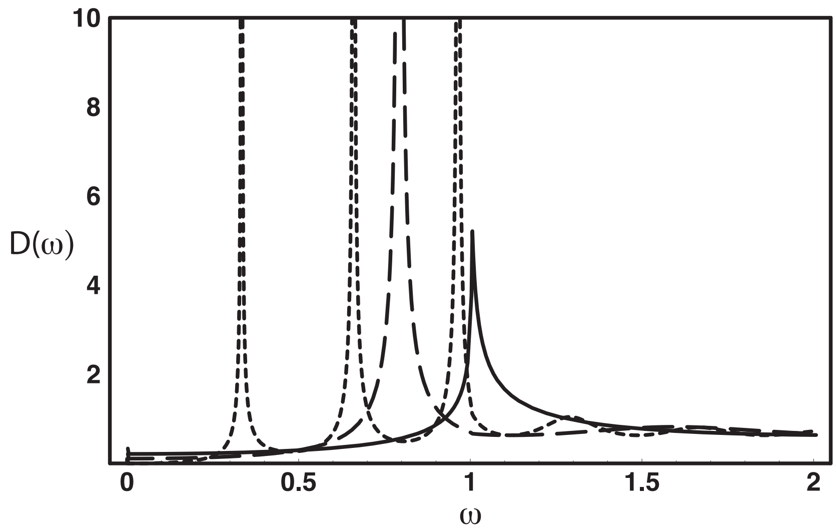


Figure 3: $D(U_0, \omega)$ for the wave equation as a function of $\omega$, $0 \leq \omega \leq 2$, for three different values of $c_0$: $c_0 = 1.02$ (solid line), $c_0 = 2$ (dashed line) and $c_0 = 6$ (dotted line).

In fact, resonances are observed also in the supercritical region, although the peaks are not pronounced. This can be explained by noting that in the supercritical regime energy is lost by radiation at infinity, so that the system experiences dissipation, which is responsible for the reduction of the peaks of the resonance curve, as in classical damped oscillators. The large reduction of the height of the peaks for $\omega > 1$, seen in Fig. 3, means that the systems is very damped, namely, that a lot of energy flows toward infinity.

### 4.1.3 Second-order solution

The PDE governing the second order terms has the same homogenous part of the one governing the first order terms (compare eq. (45) with eq. (47) and eq. (46) with eq. (48)). They differ only in the known terms, and in the fact that homogenous boundary conditions now hold. Thus, the solution can be sought in the same form as for the first-order terms. However, the known terms are much more complicated in this case, so that the actual computation of the solution is much harder. This notwithstanding, the main properties of the second order terms can be inferred directly from the governing equations (47)-(48).

The most important property is that, since the known term contains expressions proportional to $\sin(2\omega t)$ and $\cos(2\omega t)$, the second order solutions exhibit superharmonic oscillations with frequency $2\omega$, and the critical frequency, which is the boundary between the subcritical and the supercritical regimes, is $\omega = 1/2$. This implies that, for $\omega < 1/2$, the first and second order terms decay exponentially as $x \to \infty$. In the frequency interval $1/2 < \omega < 1$, the first order term still decays exponentially, but the second order term behaves like a propagating wave. Thus, for $x$ large enough, the beam experiences superharmonic oscillations, although of small (of the order of $\varepsilon^2$) amplitude, and this is the most evident consequence of the nonlinearity of the problem. Finally, for $\omega > 1$ both first and second order terms behave like propagating waves, and the harmonic behaviour is dominant.

Of course, the previous argument can be repeated for the terms of order $n$ in $\varepsilon$, $n > 2$, which exhibit superharmonic behaviour with frequency $n\omega$, and which become supercritical when the frequency overcomes the critical threshold $1/n$. However, the amplitudes of this terms are very small (of the order of $\varepsilon^n$) and are not interesting from a practical point of view.

### 4.2 Beam equation

We now turn to the beam equation (17)-(18). The transformed equations are in this case

$$c^4 \frac{\partial^2 u}{\partial t^2} + c^2 \dot{c}^2 z^2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{4} \frac{\partial^4 u}{\partial z^4} - 2c^3 \dot{c} z \frac{\partial^2 u}{\partial t \partial z} + c^2 \left(2\dot{c}^2 - c\ddot{c}\right) z \frac{\partial u}{\partial z} + c^4 = 0, \qquad z < 1 \tag{61}$$

$$c^4 \frac{\partial^2 u}{\partial t^2} + c^2 \dot{c}^2 z^2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{4} \frac{\partial^4 u}{\partial z^4} - 2c^3 \dot{c} z \frac{\partial^2 u}{\partial t \partial z} + c^2 \left(2\dot{c}^2 - c\ddot{c}\right) z \frac{\partial u}{\partial z} + c^4(1+u) = 0, \qquad z > 1 \tag{62}$$

with the boundary conditions

$$u(0,t) = U_0(1 + \varepsilon \sin \omega t) \tag{63}$$

$$\frac{\partial^2 u}{\partial z^2}(0,t) = 0 \tag{64}$$

$$u(z,t) \quad \text{bounded as } z \to \infty \tag{65}$$

and the additional continuity conditions

$$u(1^-,t) = u(1^+,t) = 0 \tag{66}$$

$$\frac{\partial u}{\partial z}(1^-,t) = \frac{\partial u}{\partial z}(1^+,t) \tag{67}$$

$$\frac{\partial^2 u}{\partial z^2}(1^-,t) = \frac{\partial^2 u}{\partial z^2}(1^+,t) \tag{68}$$

$$\frac{\partial^3 u}{\partial z^3}(1^-,t) = \frac{\partial^3 u}{\partial z^3}(1^+,t). \tag{69}$$

Next, we introduce the perturbative expansion (32)-(33) into the transformed equations (61)-(62) and obtain the usual hierarchy of equations by equating to zero the coefficients of the powers of $\varepsilon$. Again, the details of the calculations are very long and tedious, and we have carried them out with the help of a symbolic manipulation program.

To order $\varepsilon^0$ we have:

$$1 + \frac{u_0^{(IV)}(z)}{4\,c_0^4} = 0, \qquad z < 1 \tag{70}$$

$$1 + u_0(z) + \frac{u_0^{(IV)}(z)}{4\,c_0^4} = 0; \qquad z > 1 \tag{71}$$

to order $\varepsilon^1$:

$$\frac{\partial^2 u_1}{\partial t^2} + \frac{1}{4c_0^4}\frac{\partial^4 u_1}{\partial z^4} - \frac{z\,u_0'(z)\,\ddot{c}_1(t)}{c_0} - \frac{c_1(t)\,u_0^{(IV)}(z)}{{c_0}^5} = 0, \qquad z < 1 \tag{72}$$

$$\frac{\partial^2 u_1}{\partial t^2} + \frac{1}{4c_0^4}\frac{\partial^4 u_1}{\partial z^4} + u_1(z,t) - \frac{z\,u_0'(z)\,\ddot{c}_1(t)}{c_0} - \frac{c_1(t)\,u_0^{(IV)}(z)}{{c_0}^5} = 0; \quad z > 1 \tag{73}$$

to order $\varepsilon^2$:

$$\begin{aligned} &\frac{\partial^2 u_2}{\partial t^2} + \frac{1}{4c_0^4}\frac{\partial^4 u_2}{\partial z^4} + \frac{2\,z\,\dot{c}_1(t)^2\,u_0'(z)}{c_0^2} + \frac{z\,c_1(t)\,u_0'(z)\,\ddot{c}_1(t)}{c_0^2} - \frac{z\,{u_0}'(z)\,\ddot{c}_2(t)}{c_0} + \\ &\quad \frac{z^2\,\dot{c}_1(t)^2\,u_0''(z)}{{c_0}^2} + \left(\frac{16\,c_1(t)^2}{4\,{c_0}^6} - \frac{24\,{c_0}^2\,c_1(t)^2 + 16\,{c_0}^3\,c_2(t)}{16\,{c_0}^8}\right) u_0^{(IV)}(z) - \\ &\quad \frac{z\,\ddot{c}_1(t)}{c_0}\,\frac{\partial u_1}{\partial z} - \frac{2\,z\,\dot{c}_1(t)}{c_0}\,\frac{\partial^2 u_1}{\partial z \partial t} - \frac{c_1(t)}{c_0^5}\,\frac{\partial^4 u_1}{\partial z^4} = 0, \qquad z < 1 \end{aligned} \tag{74}$$

$$\begin{aligned} &\frac{\partial^2 u_2}{\partial t^2} + \frac{1}{4c_0^4}\frac{\partial^4 u_2}{\partial z^4} + u_2(z,t) + \frac{2\,z\,\dot{c}_1(t)^2\,u_0'(z)}{c_0^2} + \frac{z\,c_1(t)\,u_0'(z)\,\ddot{c}_1(t)}{c_0^2} - \\ &\quad \frac{z\,u_0'(z)\,\ddot{c}_2(t)}{c_0} + \frac{z^2\,\dot{c}_1(t)^2\,u_0''(z)}{c_0^2} + \left(\frac{16\,c_1(t)^2}{4\,{c_0}^6} - \frac{24\,{c_0}^2\,c_1(t)^2 + 16\,{c_0}^3\,c_2(t)}{16\,{c_0}^8}\right) u_0^{(IV)}(z) - \\ &\quad \frac{z\,\ddot{c}_1(t)}{c_0}\,\frac{\partial u_1}{\partial z} - \frac{2\,z\,\dot{c}_1(t)}{c_0}\,\frac{\partial^2 u_1}{\partial z \partial t} - \frac{c_1(t)}{c_0^5}\,\frac{\partial^4 u_1}{\partial z^4} = 0, \qquad z > 1. \end{aligned} \tag{75}$$

The boundary conditions associated with this hiererchy of equations are

$$\begin{array}{lll} u_0(0) = U_0, & u_0''(0) = 0, & u_0(1) = 0 \\ u_1(0,t) = U_0 \sin\omega t, & \dfrac{\partial^2 u_1}{\partial z^2}(0) = 0, & u_1(1,t) = 0 \\ u_2(0,t) = 0, & \dfrac{\partial^2 u_2}{\partial z^2}(0) = 0, & u_2(1,t) = 0 \end{array} \tag{76}$$

while the continuity conditions on the derivatives (67)-(69) have to hold at all orders.

### 4.2.1 Zero-order solution

Equations (70)-(71), with the boundary conditions for $u_0$ given in (76) are easily integrated giving

$$u_0(z) = c_0 + {c_0}^2 + \frac{2\,{c_0}^3}{3} + \frac{{c_0}^4}{6} - \frac{z^4\,{c_0}^4}{6} + z\left(-c_0 - {c_0}^2 - {c_0}^3 - \frac{{c_0}^4}{3}\right) + z^3\left(\frac{{c_0}^3}{3} + \frac{{c_0}^4}{3}\right), \qquad z < 1, \tag{77}$$

$$u_0(z) = -1 + e^{c_0 - z\,c_0}\,\left(\cos(c_0 - z\,c_0) + \sin(c_0 - z\,c_0)\,c_0\right), \qquad z > 1. \tag{78}$$

These two equations define the same function given in (26)-(27) as the static solution of the problem for the beam equation, with the identification $x = c_0 z$. The value of $c_0$ is then obtained by using the continuity conditions on the derivatives, eq. (67)-(69), and is consistent with equation (28).

### 4.2.2 First-order solution

We obtain the first order solution by following the same steps outlined in Section 4.1.2 for the wave equation. The boundary conditions for $f_{1n}(z)$ and $g_{1n}(z)$ now are

$$\begin{array}{ll} f_{11}(0) = U_0 & \\ f_{1n}(0) = 0, & n \neq 1 \\ g_{1n}(0) = 0, & \forall n \\ f_{1n}''(0) = g_{1n}''(0) = 0, & \forall n \\ f_{1n}(1) = g_{1n}(1) = 0. & \forall n \end{array}$$

and the requirements that the functions be bounded as $z \to \infty$ and that there are no waves travelling to the left have to be imposed. We show here the differential equation governing $f_{11}(z)$, which gives the only non-vanishing contribution to $u_1(z,t)$:

$$\omega^2 \left( -z + \frac{4\omega^2}{c_0} - z\,c_0 - z\,{c_0}^2 + z^3\,{c_0}^2 - \frac{z\,{c_0}^3}{3} + z^3\,{c_0}^3 - \frac{2\,z^4\,{c_0}^3}{3} \right) a_{11}$$

$$- \omega^2\, f_{11}(z) + \frac{f_{11}^{(IV)}(z)}{4\,{c_0}^4} = 0, \qquad z < 1 \tag{79}$$

$$e^{(1-z)\,c_0} \left\{ \sin[(1-z)\;c_0] \left[ 4 + z\,\omega^2\;(1-c_0) \right] - \cos[(1-z)\;c_0] \left[ z\,\omega^2\;(1+c_0) - \frac{4}{c_0} \right] \right\} a_{11} +$$

$$(1-\omega^2)\, f_{11}(z) + \frac{f_{11}^{(IV)}(z)}{4\,{c_0}^4} = 0, \quad z > 1. \tag{80}$$

Again, the solutions of equation (80) depend crucially upon $\omega$: two of the four linearly independent solutions of the associated homogeneous equation exhibit exponential behaviour ("subcritical" case) if $\omega < 1$ and oscillatory behaviour ("supercritical" case) if $\omega > 1$. As in the case of the wave equation, the threshold between subcritical and supercritical behaviour is different at different orders in $\varepsilon$; for the second-order terms, for example, the transition occurs at $\omega = 1/2$. In the subcritical case, the boundary conditions that the solution be bounded as $z \to \infty$ must be used, while in the supercritical case we must ensure that there are no travelling waves returning from infinity.

**Subcritical case** ($\omega < 1$).

As in the case of the wave equation, we find that $f_{11}$ and $a_{11}$ are the only non-vanishing contributions to the solution for $u_1(z,t)$ and $c_1(t)$; therefore we obtain

$$u_1(z,t) = f_{11}(z) \sin \omega t$$
$$c_1(t) = a_{11} \sin \omega t,$$

where $f_{11}(z)$ is the solution of equations (79)-(80) with the assigned boundary conditions; the coefficient $a_{11}$ is then determined by the continuity conditions on $f'_{11}(z)$, $f''_{11}(z)$ and $f'''_{11}(z)$ at $z = 1$. In the case of the beam equation, we couldn't find a simple expression for $a_{11}$ as a function of $U_0$ and $\omega$, as we did for the wave equation, and we had to solve numerically the algebraic system which gives $a_{11}$ as a solution.

**Supercritical case** ($\omega > 1$).

Also in this case, like for the wave equation, the solution is given by

$$u_1(z,t) = f_{11}(z) \sin \omega t + g_{11}(z) \cos \omega t$$
$$c_1(t) = a_{11} \sin \omega t + b_{11} \cos \omega t,$$

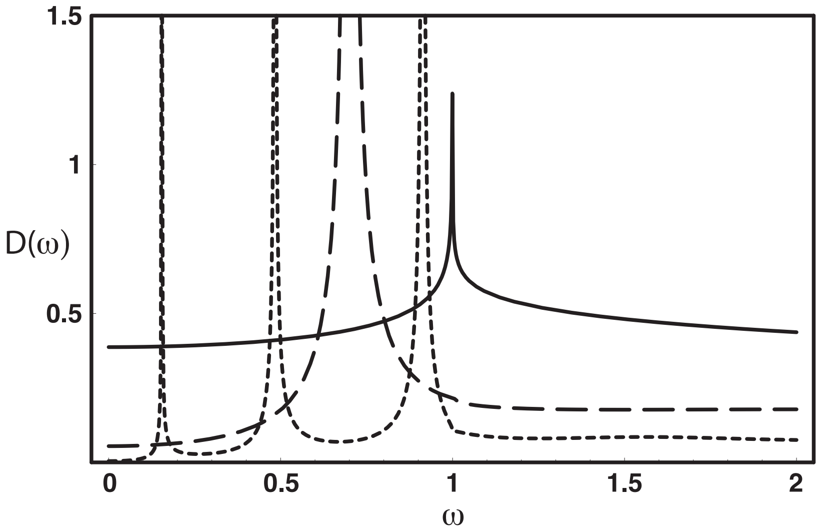


Figure 4: $D(U_0, \omega)$ for the beam equation as a function of $\omega$, $0 \leq \omega \leq 2$, for three different values of $c_0$: $c_0 = 0.5$ (solid line), $c_0 = 2$ (dashed line) and $c_0 = 6$ (dotted line).

with $a_{11}$ and $b_{11}$ given by the solution of the algebraic system obtained by matching the derivatives at $z = 1$.

The maximum elongation $\Gamma(\varepsilon)$ of the TDP is again given by $\Gamma = |a_{11}|\,\varepsilon$ in the subcritical case, and by $\Gamma = \sqrt{a_{11}^2 + b_{11}^2}\,\varepsilon$ in the supercritical case. In Figure 4, we show the amplification factor $D(U_0, \omega)$ as a function of $\omega$ for three different values of $c_0$: $c_0 = 0.5$ (solid line, corresponding to $U_0 = 0.8$), $c_0 = 2$ (dashed line, corresponding to $U_0 = 14$) and $c_0 = 6$ (dotted line, corresponding to $U_0 = 402$). The results are similar to the ones of the wave equation, with the presence of marked resonances in the subcritical region, and whose number increases at increasing $U_0$ or $c_0$.

Also in this case, there exist very weak resonances in the supercritical regime, represented by small amplitude peaks; however, due to the choice of the parameters, they are not visible in Figure 4.

#### 4.2.3 Second-order solution

The computation of the second order terms is here even harder than in the case of Section 4.1.3, but the main properties of the solution can still be understood by examining the governing equation. Nicely enough, for $\omega < 1/2$ the first and second order terms still decay exponentially as $x \to \infty$; for $1/2 < \omega < 1$ and for $x$ large enough, the beam experiences superharmonic oscillations, although of small amplitude; and for $\omega > 1$ both first and second order terms behave like propagating waves, and the harmonic behaviour is dominant.

We conclude that the wave and the beam equations share the same mechanical behaviour with respect to the problem considered here.

# 5 Conclusions and outlook

The nonlinear dynamics of semi-infinite beams and cables resting on a unilateral elastic substrate has been investigated by means of a classical perturbative approach, after an appropriate change of variables which transforms the moving boundary problem into one with fixed boundaries.

Two different regimes have been identified, one below (called subcritical) and one above (called supercritical) a certain critical excitation frequency. In the latter, energy is lost by radiation at infinity, while in the former this phenomenon does not occur. On the contrary, in the subcritical regime various resonances are observed; their number depends on the statical configuration around which the system performs nonlinear oscillations and they are absent, or better, less pronounced in the supercritical regime, due to the dissipation by radiation.

The coupling between the nonlinearity and the unboundedness of the domain is studied, and it is shown that, in the subcritical regime, far enough from the TDP in the direction of the unbounded boundary, the dominant oscillation is superharmonic, although its amplitude is orders of magnitude smaller than the amplitude of the harmonic excitation.

Finally, it has been shown that beams and cables share the previous mechanical properties, which therefore are supposed to be very general, in spite of the known different behaviour of these two mechanical systems in terms of wave propagation.

Various developments are possible and worthy. The first one is certainly the computation of the second order terms, which are expected to confirm the predictions of Sections 4.1.3 and 4.2.3 and to show the presence of other resonances. Then, it would be interesting to use more sophisticated analytical tools, such as the multiple scales method, to get a more refined solution. In this respect, we note that the problem would be particularly enticing, because both slow time scales and slow spatial scales would be required.

By the multiple scales method one can also approach the problem of nonlinear normal modes, which will allow a deep investigation of the nonlinearity of the model.

Of course, the final objective is that of considering the full J-lay problem which actually motivates this work. However, the passage to (very) large displacement is not expected to be easily solved.

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