Location and multiplicity results for multivalued functionals

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Abstract

In this paper we prove the Willem deformation lemma for continuous functionals and we extend a minmax and a multiplicity result for multivalued functionals with closed graph. This result generalize a large class of results of this type.

1 Introduction

A critical point theory for multivalued functionals with closed graph is developed by M. Frigon in the paper [6]. First we recall some definitions and results from this paper and from [3], [4]. Let (X, d) be a metric space and let $F : X \to \overline{I\!R}$ be a multivalued mapping with closed graph and nonempty values. We denote by

$$graphF = \{(u, c) \in X \times I\!R \quad | \quad c \in F(u) \quad \}.$$

The set $graphF$ is a metric space endowed with the metric

$$d_g((u, b), (v, c)) = \sqrt{d^2(u, v) + |b - c|^2}.$$

Now, we recall the definition of *weak slope* for F, see [6].
Definition 1.1 *Let $F : X \to \overline{I\!R}$ be a multivalued mapping with closed graph and let $(u, b) \in graphF$ be a point. The* **weak slope** *of F at (u, b), denoted by $|dF|(u, b)$ is the supremum of $\sigma \in [0, \infty[$ such that there exists $\delta > 0$ and a continuous function*

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2); B((u, b), \delta) \times [0, \delta] \to graphF,$$

(where $B((u, b), \delta)$ is the open ball in $graphF$ centered at (u, b) of radius δ) such that

$$(1.0a) \qquad d_g(\mathcal{H}((v, c), t), (v, c)) \le t\sqrt{1 + \sigma^2};$$

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$$(1.0b) \qquad \mathcal{H}_2((v,c),t) \leq c - \sigma t.$$

Definition 1.2 *Let (X,d) be a complete metric space and let $f : X \to I\!R$ be a continuous function and $u \in X$ a fixed element. We denote by $|df|(u)$ the supremum of the $\sigma \in [0,\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : B(u,\delta) \times [0,\delta] \to I\!R$$

such that $\forall v \in B(u,\delta)$ *for all* $t \in [0,\delta]$ *we have*

(a) $d(\mathcal{H}(v,t),v) \leq t$

(b) $f(\mathcal{H}(v,t)) \leq f(v) - \sigma t$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.
In the case where $F(u) = \{f(u)\}$ is a continuous single-valued function then $|dF|(u,f(u)) = |df|(u)$. We define the function $\mathcal{G}_F : graphF \longrightarrow I\!R$ given by $\mathcal{G}_F(u,c) = c$, where $(u,c) \in graphF$. Of course $\mathcal{G}_F$ is continuous.
Remark 1.3 For $(u,c) \in graphF$

$$|dF|(u,c) = \begin{cases} \dfrac{|d\mathcal{G}_F|(u,c)}{\sqrt{1 - |d\mathcal{G}_F|^2(u,c)}}, & |d\mathcal{G}_F|(u,c) < 1 \\ \infty\ , & |d\mathcal{G}_F|(u,c) = 1 \end{cases}$$

Definition 1.4 *a) Let $F : X \to \overline{I\!R}$ be a multivalued mapping with closed graph, and let $c \in I\!R$. We say that $u \in X$ is a critical point of F at level c, if $c \in F(u)$ and $|dF|(u,c) = 0$. The set of critical points of F at level c will be denoted by K_c. We say that c is a critical value of F if $K_c \neq \emptyset$, i.e. (u,c) is a critical element of F.*
Definition 1.5 *We say that the multivalued function $F : X \to \overline{I\!R}$ satisfies the Palais-Smale condition at level c (short $(PS)_c$), if for every sequence $(u_k) \subset X$ for which $c_n \in F(x_n)$ with $c_n \to c$ and $|dF|(u_n,c_n) \to 0$, has a convergent subsequence in X.*
Remark 1.6 The multivalued function $F : X \to \overline{I\!R}$ satisfies the condition $(PS)_c$ if and only if the function $\mathcal{G}_F$ satisfies the Palais-Smale condition at level c.
Remark 1.7 The element $(u,c) \in$ is a critical point for $\mathcal{G}_F$ if and only if u is a critical point of F.

We introduce the following notations

$$f^c = \{x \in X | \ \ f(x) \leq c\}$$

$$f_c = \{x \in X | \ \ f(x) \ \geq c\}$$

$$C_\delta := \{ \ x \in X \mid d(x,C) \leq \delta \ \}, \delta > 0.$$

To prove the Willem deformation lemma for continuous functionals, we need the following results and notions.

Definition 1.8 *Let $f : X \to I\!R \cup \{\infty\}$ be a lower semicontinuous function. We define the function*

$$\mathcal{G}_f : epi(f) \to I\!R$$

putting

$$epi(f) = \{(u,\xi) \in X \times I\!R : f(u) \leq \xi\} \quad and \quad \mathcal{G}_f(u,\xi) = \xi.$$

In the following $epi(f)$ will be endowed with the metric

$$d_{ep}((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi-\mu)^2)^{\frac{1}{2}}.$$

Of course $epi(f)$ is closed in $X \times I\!R$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u,\xi) \leq 1$ for every $(u,\xi) \in epi(f)$.

Proposition 1.9([4, Proposition 2.3]) *Let $f : X \to I\!R$ be a continuous function and let $(u,\xi) \in epi(f)$. Then*

$$|d\mathcal{G}_f|(u,\xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1+|df|(u)^2}}, & if \ \ f(u) = \xi and \ \ |df|(u) < \infty, \\ 1, & if \ \ f(u) < \xi or \ \ |df|(u) = \infty. \end{cases}$$

Theorem 1.10 ([3, Theorem 2.11]) *Let (X,d) be a complete metric space and let $f : X \longrightarrow I\!R$ be a continuous function, C a closed subset of X and $\delta,\sigma > 0$ such that*

$$d(u,C) \leq \delta \Longrightarrow |df|(u) > \sigma.$$

Then there exists a continuous map $\eta : X \times [0,\delta] \to X$ such that

1) $d(\eta(u,t),u)) \leq t,$
2) $f(\eta(u,t)) \leq f(u),$
3) $d(u,C) \geq \delta \Longrightarrow \eta(u,t) = u,$
4) $u \in C \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma t.$

Theorem 1.11 *Let (X,d) be a complete metric space, $f : X \longrightarrow I\!R$ a continuous function, C a closed subset of X, $c \in I\!R$ and $\varepsilon,\lambda > 0$. Suppose that*

(1.0) $\quad C\cap f^{c+\varepsilon'} \cap f_{c-\varepsilon'} \neq \emptyset$, *where* $\varepsilon' = \dfrac{\varepsilon \min\{\varepsilon,\lambda\}}{2\sqrt{1+\varepsilon^2}}$ *and*

(1.1) $\quad \forall u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} \Longrightarrow |df|(u) > \varepsilon.$

Then there exists a continuous map $\eta : X \times [0,1] \longrightarrow X$ such that:

a) $d(\eta(u,t),u) \leq \lambda t, \quad \forall t \in [0,1], \ \forall u \in X,$
b) $f(\eta(u,t)) \leq f(u), \quad \forall t \in [0,1], \ \forall u \in X,$
c) if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \eta(u,t) = u, \quad \forall t \in [0,1]$
d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'},$
e) $\forall \ t \in]0,1]$ *and* $\forall \ u \in f^c \cap C$ *we have* $f(\eta(u,t)) < c.$

Proof. First, we suppose that the function $f : X \longrightarrow I\!R$ is Lipschitz continuous with constant 1. We consider the set:

(1.2) $\quad C^* := \{u \in X | c - \varepsilon \le f(u) \le c + \varepsilon, d(u, C) \le \varepsilon \}$.
Obvious the set C^* is a closed subset of X, and is not empty from (1.0). We observe that $d(u, C^*) \le \varepsilon$ implies $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon) \cap C_{2\varepsilon}$.
In fact, since $|f(v) - f(u)| \le 1 \cdot d(u, v)$ for $\forall u, v \in X$ we obtain

$$-d(u, v) \le f(u) - f(v) \le d(u, v), \quad \forall v \in C^*.$$

Using this relation and the fact $d(u, C^*) \le \varepsilon$ we get

$$c - 2\varepsilon \le f(u) \le c + 2\varepsilon.$$

In a similar way, it's easy to verify that $d(u, C^*) \le \varepsilon$ implies $u \in C_{2\varepsilon}$ also.

Because $\varepsilon > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ from the above we obtain $|df|(u) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$, for all u s.t. $d(u, C^*) \le \varepsilon$.
Now we can apply Theorem 1.10, for $C := C^*, \delta := \varepsilon, \sigma := \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$. We get a continuous function $\eta' : X \times [0, \varepsilon] \longrightarrow X$ which satisfies the conditions 1)-4) from Theorem 1.10. Let $\lambda_1 := \min\{\lambda, \varepsilon\}$ and define the function $\eta : X \times [0, 1] \longrightarrow X$ by $\eta(u, t) = \eta'(u, \lambda_1 t)$. The properties a) and b) are obvious. If $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and since f is a Lipschitz function with constant 1, we have $d(u, C^*) > \varepsilon$ and using 3) from Theorem 1.10 we get $\eta(u, t) = u$.
For the proof of d) we distinguish two cases:
(1.3) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) \ge c - \varepsilon'$ it follows that $u \in C^*$, hence we have

$$f(\eta(u, 1)) = f(\eta'(u, \lambda_1)) \le f(u) - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} \le c + \varepsilon' - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} = c - \varepsilon'.$$

(1.4) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) < c - \varepsilon'$, then from b) we get

$$f(\eta(u, 1)) \le f(u) < c - \varepsilon'.$$

For the proof of e) we use also the 4) from Theorem 1.10.
Now we consider the general case. For this let $C^{**} = \{(u, \xi) \in epi(f) |\ u \in C\}$. The set $epi(f)$ is closed in $X \times I\!R$ and it follows that $epi(f)$ is a complete metric space with the metric d_{ep}. In the next we prove that for every $(u, \xi) \in epi(f)$ with $(u, \xi) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C^{**}_{2\varepsilon}$, we have $|d\mathcal{G}_f|(u, \xi) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
We distinguish two cases:
I) Let $f(u) = \xi$. In this case we have two subcases.
a) $|df|(u) < \infty$. If $(u, f(u)) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C^{**}_{2\varepsilon}$, then we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon])$ and $d_{ep}((u, f(u)), C^{**}) \le 2\varepsilon$. Since $d(u, C) \le d_{ep}((u, f(u)), C^{**}) \le 2\varepsilon$ we get $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and using (1.1) it follows that $|df|(u) > \varepsilon$.
Since $|df|(u) < \infty$ from Proposition 1.9 we have $|d\mathcal{G}_f|(u, f(u)) = \dfrac{|df|(u)}{\sqrt{1+|df|^2(u)}}$ and using the fact that the function $x \mapsto \dfrac{x}{\sqrt{1+x^2}}$ is increasing we have $|d\mathcal{G}_f|(u, f(u)) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

b) If $|df|(u) = \infty$ using Proposition 1.9 we get $|d\mathcal{G}_f|(u, f(u)) = 1 > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
II) If $f(u) < \xi$, then from Proposition 1.9 we have $|d\mathcal{G}_f|(u, \xi) = 1 > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ also.
From these we get that if $(u, \xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}^{**}$ then $|d\mathcal{G}_f|(u, \xi) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

We apply the previous step for $X := epi(f)$, $f := \mathcal{G}_f$ and $C := C^{**}$, using the fact that $\mathcal{G}_f$ is Lipschitz continuous with constant 1. Of course $C^{**} \cap \mathcal{G}_f^{c+\varepsilon'} \cap (\mathcal{G}_f)_{c-\varepsilon'} \neq \emptyset$. Then there exists a continuous mapping $\overline{\eta} := (\overline{\eta_1}, \overline{\eta_2}) : epi(f) \times [0,1] \longrightarrow epi(f)$ such that the following hold:
(1.5) $d_{ep}((\overline{\eta}(u, \xi), t), (u, \xi)) \leq \lambda t, \quad \forall\ (u, \xi) \in epi(f), \forall\ t \in [0,1]$;
(1.6) $\mathcal{G}_f(\overline{\eta}(u, \xi), t) = \overline{\eta}_2((u, \xi), t) \leq \xi = \mathcal{G}_f(u, \xi)$, for all$(u, \xi) \in epi(f)$, and $\forall\ t \in [0,1]$;
(1.7) $\overline{\eta}((u, \xi), t) = (u, \xi)$ for every $(u, \xi) \in epi(f), t \in [0,1]$ with$(u, \xi) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$;
(1.8) $\overline{\eta}(\mathcal{G}_f^{c+\varepsilon'} \cap C^{**}, 1) \subset \mathcal{G}_f^{c-\varepsilon'}$;
(1.9) $\mathcal{G}_f(\overline{\eta}((u, \xi), t)) < c$ for every $t \in]0,1]$ and $\forall\ (u, \xi) \in \mathcal{G}_f^c \cap C^{**}$.

We define the function $\eta : X \times [0,1] \longrightarrow X$ by
(1.10) $\eta(u, t) = \overline{\eta}_1((u, f(u)), t)$.
Because $\overline{\eta}$ takes its values in $epi(f)$, we have
(1.11) $f(\overline{\eta}_1((u, f(u)), t)) \leq \overline{\eta}_2((u, f(u)), t)$
From (1.5) we have:

$$d(\eta(u,t), u) = d(\overline{\eta}_1((u, f(u)), t), u) \leq$$
$$\leq [d^2((\overline{\eta}_1(u, f(u)), t), u) + (\overline{\eta}_2((u, f(u)), t) - f(u))^2]^{\frac{1}{2}} =$$
$$= d_{ep}(\overline{\eta}((u, f(u)), t), (u, f(u))) \leq \lambda t.$$

From the relations (1.6) and (1.11) we get

$$f(\eta(u,t)) = f(\overline{\eta}_1((u, f(u)), t)) \leq \overline{\eta}_2((u, f(u)), t) \leq f(u).$$

From $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$ we have $(u, f(u)) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}^{**}$. Therefore if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, then from (1.8) we get $\eta(u,t) = \overline{\eta}_1((u, f(u)), t) = u$.

If $f(u) \leq c + \varepsilon'$ then from (1.8) and (1.11) we get

$$f(\eta(u,1)) = f(\overline{\eta}_1(u, f(u)), 1) \leq \overline{\eta}_2((u, f(u)), 1) \leq c - \varepsilon'.$$

From (1.9) and (1.11) we get the relation e).

In the next we use the following remark.
Remark 1.12 If (X, d) is a metric space and A is a subset of X, then we have the following relation $d(x, A) = d(x, \overline{A})$.

Theorem 1.13 *Let (X,d) be a complete metric space and let $F : X \to \overline{I\!R}$ be a multivalued functionals with closed graph and nonempty values. Let C a subset of graphF, $c \in I\!R$ and $\lambda, \varepsilon > 0$. Suppose that $C \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$, where $\varepsilon' = \dfrac{\varepsilon \min\{\varepsilon, \lambda\}}{2\sqrt{1+\varepsilon^2}}$ and*

$$\forall u \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} \Longrightarrow |dF|(u,b) > \varepsilon, \quad \forall b \in F(u).$$

Then there exists a continuous map $\eta = (\eta_1, \eta_2) : \times[0,1] \to$ such that:

1) $d_g(\eta(u,b),t),(u,b)) \leq \lambda t$;
2) $\eta_2((u,b),t) \leq b, \quad \forall\, t \in [0,1]$ and $\forall (u,b) \in$;
3) if $u \notin \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, then $\eta((u,b),t) = (u,b)$, $\forall t \in [0,1]$;
4) $\eta(C \cap (X \times (-\infty, c+\varepsilon'], 1) \subset X \times (-\infty, c-\varepsilon']$;
5) $\forall t \in]0,1]$ and $\forall (u,b) \in C \cap (X \times (-\infty, c])$ we have $\eta_2((u,b),t) < c$.

Proof. We have $X \times I\!R$ is a complete metric space with the metric d_g defined on $(X \times I\!R)$ by $d_g((u,b),(v,c) = \sqrt{d^2(u,v) + |b-c|^2}$ for every $(u,b),(v,c) \in X \times I\!R$. Because *graphF* is a closed subset of $X \times I\!R$, we have that *graphF* is a complete metric space with d_g. We have that $|d\mathcal{G}_F|(u,b) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ and we apply the second step of Theorem 1.11 for the function $\mathcal{G}_F$ and we get the assertions of theorem.

2 Location theorem and minmax principle

In this section we prove a minmax result in the case of multivalued functionals. This result generalize a large class of minmax results for differentiable, continuous and multivalued mappings, see [7], [5], [14], [15], [8], [9]. The main tool used for the proof of this result is the generalized Willem deformation lemma for multivalued functionals, i.e. Theorem 1.13.
Let Q be a subset of *graphF*. We denote by

$$\Gamma(Q) = \{\ U \subset graphF |\ Q \subset U\ \}$$

and suppose that $U \neq \emptyset$ if $Q = \emptyset$.

Definition 2.1([6, Definition] *Let Q be a subset of graph F, and let Γ_0 be a subset of $\Gamma(Q)$. We say that Γ_0 is invariant with respect to (F, Q)-deformation, if the set $\eta(U,1) \in \Gamma_0$ for every continuous map $\eta : graphF \times [0,1] \to graphF$ such that $\eta = id$ on $graphF \times \{0\} \cup Q \times [0,1]$ and $\eta_2((u,b),t) \leq b$ for every $t \in [0,1]$ and $(u,b) \in graphF$.*
Let $A, B \subset graphF$, then in the next we use the following notations:

$$A_\delta = \{\ x \in graphF \mid d_g(x,A) \leq \delta\ \};$$

$$d(x,A) = \inf\{\ d_g(x,y) \mid y \in A\};$$

$$d(A,B) = \inf\{\ d_g(x,y) \mid x \in A, y \in B\ \};$$

$$\pi_{I\!R}(u,c)=c,\ \ \forall\ \ (u,c)\in graphF.$$

Definition 2.2 *Let A, Q be two subset of graph F, and let Γ_0 be a nonempty subset of $\Gamma(Q)$. We say that Γ_0 intersects the set A if $U\cap A\neq\emptyset$ for every $U\in\Gamma_0$.*
The main result of this section is the following which represent a generalization of some results from the papers, [7], [8], [6].

Theorem 2.3 *Let $(X,\ d)$ be a complete metric space, and $F:X\to\overline{I\!R}$ a multivalued mapping with closed graph. We assume that, there exists a closed subset A of $graphF$, and exists $Q\subset graphF$, $\Gamma_0\subset\Gamma(Q)$ a nonempty and invariant subset with respect to $(F,\ Q)$- deformation such that Γ_0 intersects A. In addition we assume that*

$$c_A=\inf_{U\in\Gamma_0}\sup\pi_{I\!R}(U\cap A)\geq\sup\pi_{I\!R}(Q)$$

with strict inequality if $d(A,Q)=0$ and the number $c=\inf\limits_{U\in\Gamma_0}\sup\pi_{I\!R}(U)$ is finite. Let $\varepsilon>0$ be a real number such that

$$\text{(2.3}a\text{)}\qquad \varepsilon<\frac{d(A,Q)}{3},\quad \textit{if}\ \ d(A,Q)>0;$$

$$\text{(2.3}b\text{)}\qquad \varepsilon<\frac{c_A-\sup\pi_{I\!R}(Q)}{2},\quad \textit{if}\ \ d(A,Q)=0;$$

$$\text{(2.3}c\text{)}\qquad \varepsilon<\frac{c-\sup_{I\!R}(Q)}{2},\quad \textit{if}\ \ c>c_A;$$

Let

$$\text{(2.3}d\text{)}\qquad E=\begin{cases}A_\varepsilon, & \textit{if}\quad c=c_A\\ \overline{\Gamma}_0\ , & \quad\textit{otherwise}\end{cases}$$

Then there exists $(u,b)\in graphF$ satisfying the following assertions
- *a)* $c-2\varepsilon\leq b\leq c+2\varepsilon$;
- *b)* $d_g((u,b),E)\leq 2\varepsilon$;
- *c)* $|dF|(u,b)\leq\varepsilon$.

Proof. First, we verify that $E\cap\mathcal{G}_F^{c+\varepsilon'}\cap(\mathcal{G}_F)_{c-\varepsilon'}\neq\emptyset$, where $\varepsilon'=\dfrac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}$.
Case I. $E=A_\varepsilon$, i.e. $c=c_A$. Let $U\in\Gamma_0$ such that $\sup\pi_{I\!R}(U)\leq c+\varepsilon'$. It's enough to prove that $A\cap U\cap(\mathcal{G}_F)_{c-\varepsilon'}\neq\emptyset$. If this is false, we have that $\sup\pi_{I\!R}(A\cap U)\leq c-\varepsilon'$, but form the definition of the c and from the hypothesis, we obtain that $c=c_A\leq\sup\pi_{I\!R}(A\cap U)$, i.e. $c\leq c-\varepsilon'$, contradiction. From this,it is clear that $A_\varepsilon\cap\mathcal{G}_F^{c+\varepsilon'}\cap(\mathcal{G}_F)_{c-\varepsilon'}\neq\emptyset$.

Case II. $E = \overline{\Gamma_0}$. We prove that $\overline{\Gamma_0} \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$. Let $U \in \Gamma_0$ as above, i.e. $\sup \pi_{I\!R}(U) \leq c+\varepsilon'$. Let us suppose, that $U \cap (\mathcal{G}_F)_{c-\varepsilon'} = \emptyset$. From this, we obtain $\sup \pi_{I\!R}(U) \leq c-\varepsilon'$. From the definition of the number c we get that $c \leq \sup \pi_{I\!R}(U)$, i.e. $c \leq c-\varepsilon'$, contradiction.
We proceed by contradiction, i.e. we assume that

$$\forall \ \ (u,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap E_{2\varepsilon} \Longrightarrow |dF|(u,b) > \varepsilon.$$

For $\lambda := \varepsilon$ we use Theorem 1.13 and we get a continuous function $\eta = (\eta_1, \eta_2) : graphF \times [0,1] \to graphF$ such that:

$$d_g(\eta((u,b),t),(u,b)) \leq \varepsilon t \tag{2.3e}$$

$$\eta_2((u,b),t) \leq b \tag{2.3f}$$

$$\forall \ \ (u,b) \notin \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap E_{2\varepsilon} \Longrightarrow \quad \eta((u,b),t) = (u,b) \tag{2.3g}$$

$$\eta(\mathcal{G}_F^{c+\varepsilon'} \cap E, 1) \subset \mathcal{G}_F^{c-\varepsilon'}. \tag{2.3h}$$

We prove that

$$\eta((u,b),t) = (u,b), \qquad \forall (u,b) \in Q, t \in [0,1]. \tag{2.3i}$$

For the proof we consider the following cases.

Case I. $E = A_\varepsilon, i.e. \ c = c_A$.
We prove that $Q \subset C_{graphF}(\mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap A_{3\varepsilon})$, where $C_{graphF}(.)$ is the complement in rapport of graph F. We assume the contrary, i.e. there exists an $(u,b) \in Q$ such that $(u,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap A_{3\varepsilon}$, then follows that $c-2\varepsilon \leq b \leq c+2\varepsilon$ and $d_g((u,b),A) \leq 3\varepsilon$. From these we have:

1. If $d(A,Q) > 0$ using the relation (2.3a) we have

$$\varepsilon < \frac{d(A,Q)}{3} \leq \frac{d(A,(u,b))}{3} \leq \varepsilon,$$

which is a contradiction.

2. If $d(A,Q) = 0$, from the relation (2.3b) we have

$$\varepsilon < \frac{c_A - \sup \pi_{I\!R}(Q)}{2} = \frac{c - \sup \pi_{I\!R}(Q)}{2} \leq \frac{c-b}{2}.$$

But from the relation $c - 2\varepsilon \leq b$ we get a contradiction.
Case II. $E = \overline{\Gamma}_0$, i.e. $c > c_A$.
We prove that

$$Q \subset C_{graphF}(\mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap (\overline{\Gamma}_0)_{2\varepsilon}).$$

We assume that there exists $(u,b) \in Q$ such that $c-2\varepsilon \le b \le c+2\varepsilon$.From this relation and relation (2.3c) we get

$$\varepsilon < \frac{c - \sup \pi_{I\!\!R}(Q)}{2} \le \frac{c-b}{2} \le \varepsilon,$$

which is a contradiction. Therefore from (2.3g) we get the relation (2.3i). It is clear, that there exists an $U_0 \subset \Gamma_0$ such that

$$(2.3j) \qquad \sup \pi_{I\!\!R}(U_0) \le c + \varepsilon'$$

Because Γ_0 is invariant with respect to (F,Q)-deformation, from the relation $\eta((u,b),0) = (u,b)$ for every $(u,b) \in graphF$ and from (2.3i), (2.3f) we have that $\eta(U_0,1) \in \Gamma_0$.

In the next we distinguish two cases.

Case I. $E = A_\varepsilon$. We have that

$$(2.3k) \qquad \eta(U_0,1) \cap A \subset \eta(U_0 \cap A_\varepsilon, 1).$$

In fact let $w_1 \in \eta(U_0,1) \cap A$. Then there exists an $w_2 \in U_0$ such that $w_1 = \eta(w_2,1)$. But $d_g(w_2,\eta(w_2,1)) \le \varepsilon$, therefore $d_g(w_2,A) \le \varepsilon$, i.e. $w_2 \in A_\varepsilon \cap U_0$. In conclusion $w_1 \in \eta(A_\varepsilon \cap U_0, 1)$. Since

$$\sup \pi_{I\!\!R}(U_0) \le c + \varepsilon'$$

and $c_A = c \le \sup \pi_{I\!\!R}(U \cap A), \ \ \forall \ U \in \Gamma_0$ using the fact that $\eta(U_0,1) \in \Gamma_0$ and the relations (2.3k), (2.3h) we have:

$$c \le \sup \pi_{I\!\!R}(\eta(U_0,1) \cap A) \le \sup \pi_{I\!\!R}(\eta(U_0 \cap A_\varepsilon, 1)) \le c - \varepsilon',$$

which is a contradiction.

Case II. $E = \overline{\Gamma}_0$. Now let $U_0 \in \Gamma_0$ as in relation (2.3j). Then from the relation (2.3h) we have $\eta(U_0,1) \subset \mathcal{G}_F^{c-\varepsilon'}$. Because $\eta(U_0,1) \in \Gamma_0$ we have $c = \inf_{U \in \Gamma_0} \sup \pi_{I\!\!R}(U) \le \sup \pi_{I\!\!R}(\eta(U_0,1)) \le c - \varepsilon'$, which is a contradiction.

As an consequence of this result we get Theorem 2.12 from [6].

Corollary 2.4 *Let X be a complete metric space, and $F : X \to \overline{I\!\!R}$ be a multi-valued mapping with closed graph. We assume that, there exists a closed subset A of $graphF$, and exists $Q \subset graphF$, and $\Gamma_0 \subset \Gamma(Q)$ nonempty and invariant with respect to (F,Q)- deformation such that Γ_0 intersects A. In addition we suppose that*

$$\inf_{U \in \Gamma_0} \sup \pi_{I\!\!R}(U \cap A) \ge \sup \pi_{I\!\!R}(Q)$$

with strict inequality if $d(A,Q) = 0$. Let $c = \inf_{U \in \Gamma_0} \sup \pi_{I\!\!R}(U)$.
If $c \in I\!\!R$, and F satisfies the condition $(PS)_c$, then we have

$$K_c \times \{c\} \cap \overline{\Gamma}_0 \ne \emptyset, \quad \textit{where} \ \ \overline{\Gamma}_0 = \overline{\cup_{U \in \Gamma_0} U}.$$

Moreover, if $c = \inf_{U \in \Gamma_0} \sup \pi_{I\!R}(U \cap A)$ *then we have* $K_c \times \{c\} \cap A \neq \emptyset$.

Proof. Theorem 2.3 implies the existence of a sequence $\{(u_n, b_n)\} \subset graphF$ such that:

Case I. $\mathcal{G}_F(u_n, b_n) \to c, d((u_n, b_n), \overline{\Gamma}_0) \to 0, |dF|(u_n, b_n) \to 0$.

Case II. $\mathcal{G}_F(u_n, b_n) \to c, d((u_n, b_n), A) \to 0, |dF|(u_n, b_n) \to 0$.

Using the condition $(PS)_c$ and the fact that the sets $\overline{\Gamma}_0$ and A are closed we get that $K_c \times \{c\} \cap \overline{\Gamma}_0 \neq \emptyset$ and $K_c \times \{c\} \cap A \neq \emptyset$, respectively.

Remark 2.5 This result generalize a large class of locations results, see [5], [7], [8].

3 Multiplicity results

Let (X, d) be a complete metric space. We consider a family of closed and nonempty subsets of X denoted by Ω.

Definition 3.1 *The function* $\Im : \Omega \to I\!N \cup \{\infty\}$ *is said be an* **index** *on* X *if satisfies the following properties:*

(i_1) $\Im(A) = 0 \iff A = \emptyset$;

(i_2) $\Im(A_1) \leq \Im(A_2)$ *if* $A_1 \subset A_2$;

(i_3) $\Im(A_1 \cup A_2) \leq \Im(A_1) + \Im(A_2)$;

(i_4) *For every continuous function* $\eta : X \times [0, 1] \to X$ *with the property* $\eta(x, 0) = x$ *for every* $x \in X$ *we have* $\Im(A) \leq \Im(\overline{\eta(A, 1)})$;

(i_5) *For some* $\delta > 0$ *we have* $\Im(A_\delta) = \Im(A)$.

In the next we use the following notations:

$$\Gamma_h = \{ \ A \subset X \mid \Im(A) \geq h \ \}, \quad h \geq 1;$$

$$\mathcal{N}_\delta(A) = \{ \ u \in X \mid d(u, A) < \delta \ \}, \text{ where } \ \delta > 0.$$

Theorem 3.2 *Let (X, d) be a complete metric space,* $f : X \to I\!R$ *be a continuous mapping which is bounded bellow and we suppose that* $\Im : \Omega \to I\!N \cup \{\infty\}$ *is an index on X. For* $1 \leq h \leq \sup\{ \ \Im(A) \mid A \subset X \ closed \ \}$ *we consider the following numbers:*

$$c_h = \inf_{A \in \Gamma_h} \sup_{x \in A} f(x). \tag{3.1a}$$

If for some $h \geq 1, m \geq 1$ *we have*

$$-\infty < c_h = \cdots = c_{h+m-1}$$

and the function f verifies the $(PS)_{c_h}$ *condition, then the following relation holds:*

$$\Im(\{ \ u \in X \mid |df|(u) = 0, \quad f(u) = c_h \ \} \geq m.$$

In particular, c_h *is a critical value for* f.

Proof. It is sufficient to adapt the classical arguments of [4, Theorem(3.7), p.90] and the properties of the index.

Theorem 3.3 *Let (X, d) be a complete metric space, $F : X \to \overline{\mathbb{R}}$ a multivalued mapping with closed and nonempty values, bounded below and $\Im$ an index on graphF. For $1 \leq h \leq \sup\{\ \Im(A) \mid A$ closed in graphF $\}$ let*

$$c_h = \inf_{A \in \Gamma_h} \sup \pi_{\mathbb{R}}(A).$$

If for some $h \geq 1, m \geq 1$ we have

$$-\infty < c_h = \cdots = c_{h+m-1}$$

and the function F verifies the Palais-Smale condition at level $c = c_h$, it holds:

$$\Im(\{\ (u,c) \in graphF \mid |dF|(u,c) = 0, \quad c \in F(u)\ \} \geq m.$$

In particular, c_h is a critical value for F.
Proof. We apply Theorem 3.2 for $f := \mathcal{G}_F$ and $X := graphF$, using the Remark 1.6 and the fact that $|dF|(u,c) = 0$ if and only if $|d\mathcal{G}_F|(u,c) = 0$.
Remark 3.4 We can consider the particular case, when the index is the Lusternik-Schnirelmann category, provided that the X is ANR. In this case, we obtain a generalization of a "classical" multiplicity result of R.S. Palais [10], M. Degiovanni- M. Marzocchi [4], J.-N. Corvellec-M. Degiovanni- M. Marzocchi [3].

References

[1] J.-N. Corvellec, *A General Approach to the Min-Max Principle*, Journal for Anal. and its Applications, Vol. 16(1997), No.2, 405-433.

[2] J.-N. Corvellec, *Morse theory for continuous functionals*, Journ. of Math. Anal. and Appl., 196(1995), 1050-1072.

[3] J.-N. Corvellec, M. Degiovanni and M. Marzocchi, *Deformation properties for continuous functionals and critical point theory*, Top. Meth. in Nonl. Anal., Vol.1,(1993), 151-171.

[4] , M. Degiovanni, M. Marzocchi, *A critical point theory for nonsmooth functionals.* Ann. Mat. Pura Appl.,**167**(1994), 73-100.

[5] G. Fang, On the existence and the classification of critical points for nonsmooth functionals, Can. J. Math. 47 (4), 1995, 684-717.

[6] M. Frigon, *On a critical point theory for multivalued functionals and applications to partial differential inclusions*, Nonlinear Analysis Theory, Methods and Applications, 31 (1998), No. 5/6, 735-753.

[7] N. Ghoussoub, *Location, multiplicity and Morse indices of min-max critical points*, Journal reine angew. Math. 417 (1991), 27-76.

[8] A. Kristály, Cs. Varga *A note on minmax results for continuous functionals*, Studia Univ."Babeş-Bolyai", Mathematica, Volume XLIII, No.3(1998),pp. 35-57.

[9] J. Mahwin, M.Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag New York, Berlin, 1989.

[10] R.S. Palais, *Lusternik-Schnirelmann theory on Banach manifolds*, Topology, vol 5(1966), 115-132.

[11] P.Pucci, J. Serrin,*Extension of the mountain pass theorem*, J. Funct. Anal. 59 (1984), 185-210.

[12] P.H. Rabinowitz, *Some minimax theorems and applications to nonlinear partial differential equations*, Nonlinear Analsis: A collection of papers in honor of Erich Rothe, Academic Press, New-York, 161-177, 1988.

[13] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMSReg. Conf. Ser. MAth. No. 65, Amer. Math. Soc., Providence, R.I., 1986.

[14] N. K. Ribarska, Ts. Y. Tsachev, M. I. Krastnov, *Speculation about mountains*, Serdica Math. Journal, 22 (1996), 341-358.

[15] N. K. Ribarska, Ts. Y. Tsachev, M. I. Krastnov, *The intrinsic mountain pass principle*, Topological Methods in Nonlinear Anal., 12, 1998, 309-322.

[16] M. Willem, *Minmax theorems*, Birkhäuser, 1995.

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