Decomposition of symplectic vector fields with respect to a fibration in lagrangian tori

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Abstract

Given a fibration of a symplectic manifold by lagrangian tori, we show that each symplectic vector field splits into two parts : the first is Hamiltonian and the second is symplectic and preserves the fibration. We then show an application of this result in the study of the regular deformations of completely integrable systems.

Introduction

We would like to begin this introduction by motivating the study of the fibrations in lagrangian tori of a symplectic manifold $(\mathcal{M}, \omega)$. Such fibrations naturally arise in the study of completely integrable systems (CI in short). These are the dynamical systems defined by a Hamiltonian $H \in C^{\infty}(\mathcal{M})$ admitting a momentum map, i.e. a set $\boldsymbol{A} = (A_1, ..., A_d) : \mathcal{M} \to \mathbb{R}^d$ of smooth functions, d being half of the dimension of $\mathcal{M}$, satisfying $\{A_j, H\} = 0$ and $\{A_j, A_k\} = 0$ for all $j, k : 1...d$, and whose differentials dA_j are linearly independent almost everywhere. Then, the Arnol'd-Mineur-Liouville Theorem [2, 10, 8] insures that in a neighbourhood of any connected component of any compact regular fiber $\boldsymbol{A}^{-1}(a)$, $a \in \mathbb{R}^d$, of the momentum map, there exists a fibration in lagrangian tori along which H is constant. These tori are thus invariant by the dynamics generated by the associated Hamiltonian vector field X_H.

Despite the "local" character of the Arnol'd-Mineur-Liouville Theorem, it is tempting to try to glue together these "local" fibrations in the case of *regular* Hamiltonians, i.e. those for which there exists, near each point of $\mathcal{M}$, a local fibration in invariant lagrangian tori. Unfortunately, this is not always possible. Some Hamiltonians do not admit any (global) fibration in lagrangian tori, and some others admit several different ones (the prototype of degenerate Hamiltonian system is the free particle moving on the sphere S^2). Nevertheless, these examples belong to the non-generic (within the class of regular

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CI Hamiltonians) class of *degenerate* Hamiltonians and one can show (see e.g. [14]) that imposing a nondegeneracy condition (like e.g. those introduced by Arnol'd [2], Kolmogorov [7], Bryuno [3] or Rüssmann [16]) insures that there exists a fibration of $\mathcal{M}$ in lagrangian tori along which H is constant, and moreover that it is unique. The genericity of nondegeneracy conditions motivates the study of fibrations in lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$.

Such a fibration actually gives rise to several natural geometric structures that we review in the first section. In particular there exists a natural process of averaging any tensor field in the direction of the fibers. This process then allows us to prove (Theorem 11) that each symplectic vector fields splits into two parts : the first is hamiltonian and the second is symplectic and preserves the fibration. As an application of this, we consider in the last section the regular deformations of CI systems and we show (Theorem 17) a *Hamiltonian* normal form for these deformations.

First, let us fix some basic notations. We denote by $\mathcal{V}(\mathcal{M})$ the space of smooth vector fields on the manifold $\mathcal{M}$. A symplectic form ω on $\mathcal{M}$ provides a isomorphism $\omega : \mathcal{V}(\mathcal{M}) \to \Omega^1(\mathcal{M})$, also denoted by ω, i.e. $\omega(X) = \omega(X, .)$ for each $X \in C^\infty(\mathcal{M})$. The inverse is denoted by $\omega^{-1} : \Omega^1(\mathcal{M}) \to \mathcal{V}(\mathcal{M})$. For each vector field, we denote by ϕ_X^t its flow at time t. Let $O \subset \mathcal{M}$ be any subset. We say that a vector field X is symplectic (resp. Hamiltonian) in $\mathcal{O}$ if its associated 1-form $\omega(X)$ is closed (resp. exact) in $\mathcal{O}$. To each Hamiltonian $H \in C^\infty(\mathcal{M})$ we can associate a vector field $X_H = -\omega^{-1}(dH)$. Now, given a fibration $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$, we say that a vector field $\tilde{X} \in \mathcal{V}(M)$ is the lift of a vector field $X \in \mathcal{V}(\mathcal{B})$ if for each $b \in \mathcal{B}$ and each $m \in \pi^{-1}(b)$ we have $\pi_*\left(\widetilde{X}_m\right) = X_b$.

1 Geometrical structures of regular CI systems

1.1 The period bundle

Let $(\mathcal{M}, \omega)$ be a symplectic manifold of dimension $2d$ and $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ a locally trivial fibration in lagrangian tori, whose fibers are denoted by $\mathcal{M}_b = \pi^{-1}(b)$, $b \in \mathcal{B}$. The tangent spaces $L_m = T_m\mathcal{M}_{\pi(m)}$ of the fibers form an integrable vector subbundle $L = \bigcup_{m\in\mathcal{M}} L_m$ of $T\mathcal{M}$. A theorem due to Weinstein [17] insures that each leaf of a lagrangian foliation (not necessarily a fibration) is naturally endowed with an affine structure. This affine structure can actually be expressed in a very convenient way (see e.g. [18]) in terms of a linear connection ∇ on the leaf, as follows :

Proposition 1. *Let $\mathcal{N}$ be a leaf of a lagrangian foliation L. Let the operator ∇ : $\mathcal{V}(\mathcal{N}) \times \mathcal{V}(\mathcal{N}) \to \mathcal{V}(\mathcal{N})$ be defined by*

$$\nabla_X Y = \omega^{-1}\left(\widetilde{X} \lrcorner d\left(\widetilde{Y} \lrcorner \omega\right)\right),$$

where $\widetilde{X} \in \Gamma(L)$ and $\widetilde{Y} \in \Gamma(L)$ extend X and Y in $\mathcal{V}(\mathcal{M})$ and are everywhere tangent to L. Then ∇ defines a torsion-free and flat connection on $\mathcal{N}$.

Accordingly, each fiber $\mathcal{M}_b$ is endowed with such a torsion-free and flat connection. Moreover, since the foliation actually defines a fibration, the holonomy of ∇ must vanish. Indeed, for each $b \in \mathcal{B}$, any set of smooth functions $f_1, ..., f_d \in C^\infty(\mathcal{B})$ whose differentials df_j are linearly independent near b, provides d Hamiltonian vector fields $X_{f_1\circ\pi}, ..., X_{f_d\circ\pi} \in \mathcal{V}(\mathcal{M})$ everywhere tangent to the fibers, parallel on each fiber and linearly independent in a neighbourhood of $\mathcal{M}_b$. They thus form a global parallel frame on $\mathcal{M}_b$, implying that the holonomy of ∇ vanishes and that each fiber $\mathcal{M}_b$ is endowed with a structure of a standard[1] affine torus. Actually, if we denote by $\mathcal{V}_\nabla(\mathcal{M}_b)$ the d-dimensional vector space composed of parallel vector fields, the argument above shows also that the union $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ is naturally endowed with a structure of a vector bundle over $\mathcal{B}$. In the sequel, the following characterization for the space of sections of the previous bundle will be useful.

Proposition 2. *A vector field $X \in \mathcal{V}(\mathcal{M})$ is vertical and parallel on each fiber if and only if its associated 1-form $\omega(X)$ is a pull-back, i.e*

$$X \in \Gamma\left(\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right) \Longleftrightarrow \omega(X) \in \pi^*\left(\Omega^1(\mathcal{B})\right).$$

Proof. Let $\alpha = \omega(X)$ be the associated 1-form and $L = \bigcup_m L_m$ the vertical lagrangian foliation tangent to the fibers. It is a well-know fact that α is an element of $\pi^*\left(\Omega^1(\mathcal{B})\right)$ if and only if both restrictions $\alpha|_L$ and $d\alpha|_L$ vanish. Since the foliation is lagrangian, the 1-form $\alpha|_L$ vanishes if and only if X is vertical. Moreover, it follows from the definition of the connection ∇ on the fibers, that X is parallel if and only if $d\alpha|_L$ vanishes. $\square$

Now, each fiber $\mathcal{M}_b$ is isomorphic to the standard torus $\mathbb{T}^d$. Thus, among the parallel vector fields on $\mathcal{M}_b$, we can consider those whose dynamics is 1-periodic. We denote this set by

$$\Lambda_b = \left\{X \in \mathcal{V}_\nabla(\mathcal{M}_b) \mid \phi_X^1 = \mathbb{I}\right\}.$$

This discrete subset can easily be shown to be a lattice in $\mathcal{V}_\nabla(\mathcal{M}_b)$. We call it the *period lattice*. We will show that the union $\Lambda = \bigcup_{b\in\mathcal{B}} \Lambda_b$ is a smooth lattice subbundle of $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ and we call it the *period bundle*. One way to proceed is to construct explicitly smooth sections of $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ which are 1-periodic. Such sections can be constructed as Hamiltonian vector fields with Hamiltonian given by the following lemma.

Lemma 3. *Let $\mathcal{O} \subset \mathcal{B}$ be an open set and θ a symplectic potential in $\tilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$. Let $b \to \gamma(b)$ be a family of cycles depending smoothly on b and such that $\gamma(b) \subset \mathcal{M}_b$ for all $b \in \mathcal{O}$. Let the function $\xi \in C^\infty(\mathcal{B})$ be defined by*

$$\xi(b) = \int_{\gamma(b)} \theta.$$

[1] Here, "standard" means holonomy-free. We recall that on the torus $\mathbb{T}^d$ there exist also exotic affine structures with non-zero holonomy, such as Nagano-Yagi's one [12]. Some authors have [4, 5] shown that such affine structures can occur on the leaves of certain lagrangian foliations. Such foliations do of course not define a fibration.

Then, the vector field $X_{\xi\circ\pi}$ associated to the Hamiltonian $\xi\circ\pi$ is vertical, parallel and 1-periodic on each torus $\mathcal{M}_b$ with $b\in\mathcal{O}$. Moreover, for all $b\in\mathcal{O}$, its trajectories on $\mathcal{M}_b$ are homotopic to the cycle $\gamma(b)$.

Such a function ξ is called an *action*. We can then show the following.

Theorem 4. *The period bundle Λ is a smooth lattice subbundle of $\bigcup_{b\in\mathcal{B}}\mathcal{V}_\nabla(\mathcal{M}_b)$. Moreover, for any contractible subset $\mathcal{O}\subset\mathcal{B}$, all smooth local sections $X\in\Gamma(\mathcal{O},\Lambda)$ are Hamiltonian in $\tilde{\mathcal{O}}=\pi^{-1}(\mathcal{O})$.*

Proof. First, the fibration being locally trivial, we can find, in a contractible subset $\mathcal{O}\subset\mathcal{B}$, d smooth families of cycles $\gamma_j(b)$, $j=1..d$, forming for each $b\in\mathcal{O}$ a basis of $\pi_1(\mathcal{M}_b)$. On the other hand, a theorem due to Weinstein [17] implies that there exists a symplectic potential in a neighbourhood of a fiber $\mathcal{M}_b$. We can always choose a smaller $\mathcal{O}$ such that there exist a symplectic potential θ in $\tilde{\mathcal{O}}=\pi^{-1}(\mathcal{O})$. For each $j=1..d$, let $\xi_j\in C^\infty(\mathcal{B})$ be the action function of the previous lemma. Then, the Hamiltonian vector fields $X_j=X_{\xi_j\circ\pi}$ are in the lattice Λ_b for each $b\in\mathcal{O}$. Moreover, these are primitive elements of the lattice since their trajectories are homotopic to the cycles $\gamma_j(b)$ which form a basis of $\pi_1(\mathcal{M}_b)$. For the same reason, they are linearly independent. The vector fields X_j thus form a smooth family of basis of Λ_b for all $b\in\mathcal{O}$. This shows the first part of the theorem. The second part follows from the fact that each local section $X\in\Gamma(\mathcal{O},\Lambda)$ decomposes into $X=\Sigma_j c_j X_j$, where the coefficients c_j are integer constant. This implies that X is Hamiltonian in $\tilde{\mathcal{O}}$. ☐

This theorem implies the existence of a natural integer flat connection on the vector bundle $\bigcup_{b\in\mathcal{B}}\mathcal{V}_\nabla(\mathcal{M}_b)$ since the lattices Λ_b provides a way to relate the spaces $\mathcal{V}_\nabla(\mathcal{M}_b)$ for neighbouring b. This connection, may have non-vanishing holonomy. We call it *monodromy*, since it coincides obviously with the monodromy of the fibration in tori (without having regards to the symplectic structure and to the fact that the tori are lagrangian).[2]

1.2 The torus action bundle

Our discussion so far shows that given a fibration in lagrangian tori $\mathcal{M}\xrightarrow{\pi}\mathcal{B}$, there exists a natural associated torus bundle acting on it. Indeed, for each $b\in\mathcal{B}$, the quotient

$$\mathcal{G}_b=\mathcal{V}_\nabla(\mathcal{M}_b)/\Lambda_b$$

is a Lie group isomorphic to the torus $\mathbb{T}^d$. This isomorphism is not canonical, but it can be realised by choosing a basis of Λ_b. We will denote the elements of $\mathcal{G}_b$ by $[X_b]$, with $X_b\in\mathcal{V}_\nabla(\mathcal{M}_b)$, since they are equivalence classes. Taking the union over all b, we get a torus bundle $\mathcal{G}=\bigcup_{b\in\mathcal{B}}\mathcal{G}_b$. It is a smooth bundle since the period bundle Λ is so. We stress the fact that $\mathcal{G}$ is in general not a principal bundle since there might not exist any global action of $\mathbb{T}^d$ on $\mathcal{G}$, because of the presence of monodromy, which precisely prevents us from choosing a global

[2] Moreover, the symplectic form ω provides an isomorphism between the sections of $\bigcup_{b\in\mathcal{B}}\mathcal{V}_\nabla(\mathcal{M}_b)$ and those of $T^*\mathcal{B}$. This gives the base space $\mathcal{B}$ a natural structure of an affine space, as was discovered by Duistermaat [6].

basis of Λ. On the other hand, there exists a distinguished global section, since each fiber is a group with a well-defined identity element.

Although we cannot apply the general theory of connections on principal bundles, there is a natural way to speak about local parallel sections of $\mathcal{G}$ over a subset $\mathcal{O} \subset \mathcal{B}$. These sections are simply local sections $b \to [X_b]$ of $\mathcal{G}$, with $b \to X_b$ being a smooth local parallel section of $\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$. We denote the set of local parallel sections by $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$.

Proposition 5. *For each contractible subset $\mathcal{O} \subset \mathcal{B}$, the space $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ is a Lie group isomorphic to the torus $\mathbb{T}^d$.*

Proof. If $\mathcal{O}$ is contractible, then the monodromy vanishes in $\mathcal{O}$ and there exist local sections $X_1, ..., X_d \in \Gamma(\mathcal{O}, \Lambda)$ with $\{X_j(b)\}$ generating the lattice Λ_b at each $b \in \mathcal{O}$. To each element $(t_1, ...t_d) \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, we associate $[X] = [t_1X_1 + ... + t_dX_d] \in \Gamma_\nabla(\mathcal{O}, \mathcal{G})$. One easily verifies that this provides an isomorphism. $\square$

For each b, the group $\mathcal{G}_b$ naturally acts on $\mathcal{M}_b$ in the following way.

$$\begin{array}{rcl} \mathcal{G}_b \times \mathcal{M}_b & \to & \mathcal{M}_b \\ ([X_b], m) & \to & [X_b](m) = \phi^1_{X_b}(m), \end{array}$$

where $X_b \in \mathcal{V}_\nabla(\mathcal{M}_b)$ is a representative of the class $[X_b]$. One can see easily that this action is commutative, free, transitive and affine with respect to Weinstein's connection on $\mathcal{M}_b$. Now, given any section $g \in \Gamma(\mathcal{G})$, its restriction $g|_\mathcal{O}$ to any contractible subset $\mathcal{O} \subset \mathcal{B}$ is of the form $g|_\mathcal{O} = [X]$, where $X \in \Gamma\left(\mathcal{O}, \bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$. We can then extend the previous fiberwise action of the $\mathcal{G}_b$ to a vertical action of the sections of the toric bundle $\mathcal{G}$ on $\mathcal{M}$ by

$$\begin{array}{rcl} \Gamma(\mathcal{G}) \times \mathcal{M} & \to & \mathcal{M} \\ (g, m) & \to & [X](m) = \phi^1_X(m), \end{array}$$

where $X \in \Gamma\left(\mathcal{O}, \bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$ for any contractible neighbourhood $\mathcal{O}$ of $b = \pi(m)$. This is well-defined since another choice X' of the representative class of $[X]$ would differ from X only by an element of $\Gamma(\mathcal{O}, \Lambda)$ which would provide $\phi^1_{X'-X} = \mathbb{I}$. This action naturally inherits the properties of the fiberwise action, and an additional property arises when we restrict ourselves to the parallel sections of $\mathcal{G}$.

Proposition 6. *For any contractible subset $\mathcal{O} \subset \mathcal{B}$, $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ acts vertically on $\mathcal{M}$ in a symplectic way.*

We call this action the *toric action* of $\mathcal{G}$ on $\mathcal{M}$. Even if this action is local, it provides a way to average any tensor field on $\mathcal{M}$. Indeed, according to Proposition 5, $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ is a compact Lie group provided $\mathcal{O} \subset \mathcal{B}$ is simply connected. It is thus endowed with its Haar measure $\mu_\mathcal{G}$ and for any tensor field T of any type on $\mathcal{M}$, we can define its *vertical average* $\langle T\rangle$ in the following way. For each $m \in \mathcal{M}$, we set

$$\langle T\rangle_m = \int_{\Gamma_\nabla(\mathcal{O},\mathcal{G})} \left(\phi^1_X\right)_*(T)\, d\mu_\mathcal{G},$$

where $\mathcal{O} \subset \mathcal{B}$ is any contractible neighbourhood of $b = \pi(m)$. We can check that the definition does not depend on the choice of $\mathcal{O}$. Choosing a basis $X_1, ..., X_d$ of $\Gamma(\mathcal{O}, \Lambda)$ provides an explicit expression for the averaged tensor, namely

$$\langle T \rangle_m = \int_0^1 dt_1 ... \int_0^1 dt_d \left(\phi_{X_1}^{t_1}\right)_* \circ ... \circ \left(\phi_{X_d}^{t_d}\right)_* (T),$$

which is also independent of the choice of the basis.

A tensor field T is called *invariant under the toric action of* $\mathcal{G}$, or simply $\mathcal{G}$-*invariant*, if for each local parallel section $X \in \Gamma_\nabla(\mathcal{O}, \mathcal{G})$ we have $\left(\phi_X^1\right)_* (T) = T$, or equivalently $\mathcal{L}_X T = 0$. The following properties can be proved in a straightforward way.

Proposition 7. *According to these definitions, we have the following basic properties :*

1. *T is $\mathcal{G}$-invariant if and only if $\langle T \rangle = T$.*
2. $\langle \langle T \rangle \rangle = \langle T \rangle$.
3. *Each p-form $\alpha \in \Omega^p(\mathcal{M})$ verifies $\langle d\alpha \rangle = d \langle \alpha \rangle$.*
4. *Let T and S be two tensor fields. If T is $\mathcal{G}$-invariant, then the contraction $T \lrcorner S$ with respect to any two indices verifies $\langle T \lrcorner S \rangle = T \lrcorner \langle S \rangle$.*
5. *In particular, if $X \in \mathcal{V}(\mathcal{M})$ is a vector field and $\alpha = \omega(X)$ its associated 1-form, then we have $\omega(\langle \alpha \rangle) = \langle X \rangle$.*

2 Decomposition of symplectic vector fields

The averaging process presented in the previous section provides a way to decompose any symplectic vector field into the sum of a Hamiltonian vector field and a symplectic vector field preserving the fibration. The key step is the following lemma.

Lemma 8. *If α is a closed 1-form on $\mathcal{M}$ whose vertical average vanishes, then it is exact. Moreover, one can choose the primitive $f \in C^\infty(\mathcal{M})$, $\alpha = df$, with the property $\langle f \rangle = 0$.*

Proof. Let us work locally in a contractible subset $\mathcal{O} \subset \mathcal{B}$. There exists a basis $(X_1, \cdots, X_d)$ of $\Gamma(\mathcal{O}, \Lambda)$. Choosing an "initial point" $m(b)$ depending smoothly on $b \in \mathcal{O}$, i.e a smooth section of the restricted bundle $\pi^{-1}(\mathcal{O}) \xrightarrow{\pi} \mathcal{O}$, let us consider the smooth family of cycles $\gamma_j(b)$ consisting of the orbits $t \to \phi_{X_j}^t(m(b))$. The homology classes $[\gamma_j(b)]$ form for each $b \in \mathcal{O}$ a basis of $H_1(\mathcal{M}_b)$. On the other hand, since the fibration $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ is locally trivial and $\mathcal{O}$ is contractible, the classes $[\gamma_j(b)]$ form a basis of the homology of $\tilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$.

We then show that for each $j = 1..d$ and each $b \in \mathcal{O}$, one has $\int_{\gamma_j(b)} \langle \alpha \rangle = \int_{\gamma_j(b)} \alpha$. Indeed, one has

$$\begin{aligned}\int_{\gamma_j(b)} \langle \alpha \rangle &= \int_0^1 dt \; \langle \alpha \rangle (X_j) \circ \phi^t_{X_j} (m(b)) \\ &= \int_0^1 dt \, X_j \lrcorner \left(\phi^{-t}_{X_j} \right)_* \langle \alpha \rangle .\end{aligned}$$

Moreover, expressing the average $\langle \alpha \rangle$ in terms of the generators X_j, one obtains

$$\int_{\gamma_j(b)} \langle \alpha \rangle = \int_0^1 dt_1 ... \int_0^1 dt_d \int_0^1 dt \, \left(\phi^{t_1}_{X_1} \right)_* \circ \cdots \circ \widehat{\left(\phi^{t_j}_{X_j} \right)_*} \circ \cdots \circ \left(\phi^{t_d}_{X_d} \right)_* \left(X_j \lrcorner \left(\phi^{t_j - t}_{X_j} \right)_* \alpha \right),$$

where the entry below $\widehat{\ }$ has been omitted. Then, we check with a trivial change of variable that

$$\int_0^1 dt_j \int_0^1 dt \left(X_j \lrcorner \left(\phi^{t_j - t}_{X_j} \right)_* \alpha \right) = \int_{\gamma_j(b)} \alpha .$$

This implies that $\int_{\gamma_j(b)} \langle \alpha \rangle = \int_{\gamma_j(b)} \alpha$.

Finally, the hypothesis $\langle \alpha \rangle = 0$ yields $\int_{\gamma_j(b)} \alpha = 0$, where the classes $[\gamma_j(b)]$ form a basis of the homology of $\tilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$, as shown before. Since α is closed, this implies that α is exact. Thus, there exists a function $f \in C^\infty \left(\tilde{\mathcal{O}} \right)$ such that $\alpha = df$ in $\tilde{\mathcal{O}}$. This function is unique up to a constant. On the other hand, we deduce from the property $\langle df \rangle = d \langle f \rangle$ and the hypothesis $\langle \alpha \rangle = 0$ that $\langle f \rangle$ is a constant function. This allows us to choose the primitive f in an unique way, requiring that $\langle f \rangle = 0$. This criterion is independent of the choice of the basis $(X_1, ..., X_d)$ and thus allows us to find a primitive f of α globally defined on $\mathcal{M}$. $\square$

This lemma has the following corollary.

Proposition 9. *If X is a symplectic vector field with vanishing vertical average, i.e $\langle X \rangle = 0$, then X is Hamiltonian and*

$$X = X_H \text{ with } \langle H \rangle = 0.$$

Proof. Indeed, let $\alpha = \omega(X, .)$ be the closed 1-form associated with X. According to Proposition 7, $\langle X \rangle = 0$ if and only if $\langle \alpha \rangle = 0$. The previous lemma then implies that α is exact $\alpha = dF$, with $\langle F \rangle = 0$, i.e X is Hamiltonian $X = X_H$, with $H = -F$. $\square$

Symplectic lifted vector fields and $\mathcal{G}$-invariant ones are related as shown in the folloing proposition.

Proposition 10. *If $X \in \mathcal{V}(\mathcal{M})$ is a $\mathcal{G}$-invariant vector field, then it is a lift of a vector field $Y \in \mathcal{V}(\mathcal{B})$.*

If $\tilde{Y} \in \mathcal{V}(\mathcal{M})$ is a symplectic lift of a vector field $Y \in \mathcal{V}(\mathcal{B})$, then it is $\mathcal{G}$-invariant.

Proof. Indeed, for each b let $\mathcal{O} \subset \mathcal{B}$ be a contractible neighbourhood of $b \in \mathcal{B}$. Since the toric action of $\mathcal{G}$ is transitive, then for each points m and m' belonging to the fiber $\mathcal{M}_b$, there exists a $[Z] \in \Gamma_\nabla(\mathcal{G}, \mathcal{O})$ such that $m' = \phi_Z^1(m)$. On the other hand, the fact that X is $\mathcal{G}$-invariant implies $X_{m'} = \left(\phi_Z^1\right)_* X_m$. Then, using $\pi \circ \phi_Z^1 = \pi$, one sees that $\pi_* X_{m'} = \pi_* \left(\phi_Z^1\right)_* X_m$ and thus $\pi_* X_{m'} = \pi_* X_m$. This proves that $X|_{\mathcal{M}_b}$ is a lift of the tangent vector $\pi_* X_m \in T_b\mathcal{B}$. Finally, it follows from the smoothness of X that it is a lift of a vector field on $\mathcal{B}$.

We postpone the proof of the second statement to the next section where we prove it in the slightly more general case of time-dependent vector fields (Proposition 15). □

We now give the announced theorem of decomposition of symplectic vector fields.

Theorem 11. *Each symplectic vector field $X \in \mathcal{V}(\mathcal{M})$ can be written in an unique way*

$$X = X_1 + X_2,$$

where

- *X_1 is a Hamiltonian vector field, $X_1 = X_A$, with $\langle A \rangle = 0$, where $\langle A \rangle$ is the vertical average of the Hamiltonian A.*
- *X_2 is symplectic and is the lift to $\mathcal{M}$ of a vector field on $\mathcal{B}$.*

Moreover, X_2 is simply the vertical average of X, i.e. $X_2 = \langle X \rangle$.

Proof. Let $\alpha = \omega(X, .)$ be the 1-form associated with X, which is closed since X is symplectic. Let $\alpha_2 = \langle \alpha \rangle$ be the vertical average of α and let $\alpha_1 = \alpha - \alpha_2$. The 1-forms α_1 and α_2 are closed since $d\langle \alpha \rangle = \langle d\alpha \rangle$. Thus, the vector fields X_1 and X_2, associated with α_1 and α_2, are symplectic. On the other hand, one has $\langle \alpha_1 \rangle = 0$ and thus $\langle X_1 \rangle = 0$. According to Proposition 9, this implies that X_1 is Hamiltonian, $X_1 = X_A$, with $\langle A \rangle = 0$. Finally, $\langle \alpha_2 \rangle = \alpha_2$ implies that $\langle X_2 \rangle = X_2$. By Proposition 10, it is a lift of a vector field on $\mathcal{B}$.

Moreover, the decomposition $X = X_1 + X_2$ is the unique one of this type. Indeed, suppose that there is a second decomposition $X = X_1' + X_2'$ with the same properties. Taking the vertical average of both expressions, we obtain $\langle X_1 + X_2 \rangle = \left\langle X_1' + X_2' \right\rangle$ and thus $\langle X_2 \rangle = \left\langle X_2' \right\rangle$. Now, by Proposition 10, both X_2 and X_2' are $\mathcal{G}$-invariant. It follows that $X_2 = X_2'$ and thus $X_1 = X_1'$. □

With respect to the fibration, the X_2 part is "trivial" since its flow preserves the fibration and the X_1 part is Hamiltonian. We stress the fact that this result still holds in the presence of monodromy. This theorem is used in the sequel to show a normal form theorem for regular deformations of CI systems.

3 Application : deformations of CI systems

3.1 Regular deformations of completely integrable systems

Let $\left(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ be a *regular CI system* composed of a fibration in lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ and a Hamiltonian $H_0 \in C^\infty(\mathcal{M})$ constant along the fibers. It is well-known since Poincaré's work [13] that adding a small perturbation εK to the CI Hamiltonian H_0 will destroy its CI character and yield chaotic behaviours. Nevertheless, it is important to investigate the space of all CI Hamiltonians, since these are the starting point of any perturbation theory, like the celebrated K.A.M. Theory [7, 1, 11] which actually tells us that one can say a lot about the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon K$ when ε is small.

A first step towards the understanding of the space of all CI systems, is to restrict ourselves to regular deformations of regular CI hamiltonians, i.e smooth families of Hamiltonians H_ε which are CI and regular for each ε. At this point, we would like to stress the fact that this does *not* imply that H_ε is constant along the fibers of a family of $\mathcal{M} \xrightarrow{\pi_\varepsilon} \mathcal{B}$ *depending smoothly on* ε. Nevertheless, we conjecture that is is true for the generic class of *non-degenerate* Hamiltonians. We refer to [14] for a review of different nondegeneracy conditions and we will now restrict our study to the following class of deformations.

Definition 12. Let $\left(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ be a regular CI system and let $H_\varepsilon \in C^\infty(\mathcal{M})$ be a smooth family of Hamiltonians. We say that H_ε is a *regular deformation* of H_0 if there exist a smooth family of functions $I_\varepsilon \in \pi^*\left(C^\infty(\mathcal{B})\right)$, with $I_0 = H_0$, and a smooth family of symplectomorphisms $\phi^\varepsilon : \mathcal{M} \to \mathcal{M}$, with $\phi^0 = \mathbb{I}$, such that

$$H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon$$

for all ε.

For our purposes, we will need to work now with time-dependent vector fields since each smooth family of diffeomorphisms ϕ^ε, with $\phi^0 = \mathbb{I}$, is the flow at time ε of the time-dependent vector field X_ε defined by

$$\frac{d\left(f \circ \phi^\varepsilon(m)\right)}{d\varepsilon} = X_\varepsilon(f) \circ \phi^\varepsilon(m)$$

for each smooth function $f \in C^\infty(\mathcal{M})$ and each point $m \in \mathcal{M}$. We denote this flow by $\phi^\varepsilon_{X_\varepsilon}$. In all the following, all the considered family ϕ^ε of diffeomorphisms will implicitly depend smoothly on ε and satisfy $\phi^0 = \mathbb{I}$. We refer e.g. to [9] for a review of the properties of time-dependent vector fields.

3.2 Normal form for regular deformations

The aim of this section is to show Theorem 17 which insures that, by changing the function I_ε, one may assume that ϕ^ε is a Hamiltonian flow. This result is based on Theorem 16 which states that any family of symplectomorphisms ϕ^ε can be written as the composition of a Hamiltonian flow with a family of fiber-preserving symplectomorphisms. Let us first define precisely these two notions.

Definition 13. A family of symplectomorphisms ϕ^ε is called *Hamiltonian* if its vector field X_ε is Hamiltonian, $X_\varepsilon = X_{A_\varepsilon}$, with $A_\varepsilon \in C^\infty(\mathcal{M})$ depending smoothly on ε.

Definition 14. A family of diffeomorphisms $\phi^\varepsilon : \mathcal{M} \to \mathcal{M}$ is called *fiber-preserving* if there exists a family of diffeomorphisms on the base space $\varphi^\varepsilon : \mathcal{B} \to \mathcal{B}$ such that

$$\pi \circ \phi^\varepsilon = \varphi^\varepsilon \circ \pi.$$

We say that ϕ^ε is *vertical* whenever $\varphi^\varepsilon = \mathbb{I}$ for all ε.

Whenever a vector field on $\mathcal{M}$ is both symplectic and a lift of a vector field on $\mathcal{B}$, then we have the following property.

Proposition 15. *If $\tilde{Y}_\varepsilon \in \mathcal{V}(\mathcal{M})$ is symplectic for each ε and is a lift of a time-dependent vector field $Y_\varepsilon \in \mathcal{V}(\mathcal{B})$, then it is $\mathcal{G}$-invariant and for each tensor field T one has*

$$\left\langle \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_* T \right\rangle = \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_* \langle T \rangle .$$

Proof. Let denote by $\phi^\varepsilon = \phi^\varepsilon_{\tilde{Y}_\varepsilon}$ the flow of $\tilde{Y}_\varepsilon$. This flow is fiber-preserving and thus verifies $\pi \circ \phi^\varepsilon = \varphi^\varepsilon \circ \pi$ with $\varphi^\varepsilon : \mathcal{B} \to \mathcal{B}$ a family of diffeomorphisms. One can easily show that φ^ε is actually the flow of Y_ε.

First of all, for each vertical and parallel vector field $X \in \Gamma\left(\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$, one has $\phi^\varepsilon_* X \in \Gamma\left(\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$. Indeed, according to Proposition 2, $\phi^\varepsilon_* X$ is vertical and parallel if and only if the 1-form $\omega(\phi^\varepsilon_* X)$ is a pull-back. Now, one has $\omega(\phi^\varepsilon_* X) = \left((\phi^\varepsilon)^{-1}\right)^* (\omega(X))$ since ϕ^ε is symplectic for each ε. On the other hand, $\omega(X) = \pi^* \beta$ with $\beta \in \Omega^1(\mathcal{B})$, since by hypothesis X is vertical and parallel. Thus, one has

$$\omega(\phi^\varepsilon_* X) = \left((\phi^\varepsilon)^{-1}\right)^* \pi^* \beta = \pi^* \left((\varphi^\varepsilon)^{-1}\right)^* \beta.$$

This proves that $\omega(\phi^\varepsilon_* X)$ is a pull-back and thus $\phi^\varepsilon_* X$ is vertical and parallel.

If in addition $X \in \Gamma(\Lambda, \mathcal{O})$, with $\mathcal{O} \subset \mathcal{B}$ a subset, i.e. X is 1-periodic in $\pi^{-1}(\mathcal{O})$, then so is $\phi^\varepsilon_* X$ in $\phi^\varepsilon\left(\pi^{-1}(\mathcal{O})\right)$. Now, the smooth bundle Λ has discrete fibers and $\phi^\varepsilon_* X$ depends smoothly on ε. This implies that for all ε, one has $\phi^\varepsilon_* X = \phi^{\varepsilon=0}_* X$ and thus $\phi^\varepsilon_* X = X$. Then, the derivative with respect to ε shows that $\left[\tilde{Y}, X\right] = 0$, i.e. $\tilde{Y}$ is $\mathcal{G}$-invariant. By linearity, this is true as well for all $X \in \Gamma\left(\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$.

Therefore, for each $X \in \Gamma\left(\bigcup_{b\in\mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)\right)$ and each ε, ϕ^ε commutes with the flow ϕ^t_X. This implies that ϕ^ε commutes with the toric action of $\mathcal{G}$ and thus with the averaging process, i.e.

$$\left\langle \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_* T \right\rangle = \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_* \langle T \rangle$$

for any tensor field T. □

We can now give the following decomposition theorem for families of symplectomorphisms.

Theorem 16. *Each family of symplectomorphisms ϕ^ε decomposes in a unique way as follows :*

$$\phi^\varepsilon = \Phi^\varepsilon \circ \phi^\varepsilon_{Z_\varepsilon},$$

where

- *Φ^ε is a fiber-preserving family of symplectomorphisms.*
- *$Z_\varepsilon = X_{G_\varepsilon}$ is an time-dependent Hamiltonian vector field with $\langle G_\varepsilon \rangle = 0$.*

Moreover, the vector field of Φ^ε equals to the average $\langle X_\varepsilon \rangle$, where X_ε is the vector field of ϕ^ε.

Proof. Let X_ε be the vector field of ϕ^ε. Theorem 11 insures that for each ε, X_ε decomposes into $X_\varepsilon = \tilde{Y}_\varepsilon + W_\varepsilon$, where $\tilde{Y}_\varepsilon$ is a lift of a vector field $Y_\varepsilon \in \mathcal{V}(\mathcal{B})$ and W_ε is Hamiltonian. Moreover, by looking more carefully at the proof of Theorem 11, one can easily check that $\tilde{Y}_\varepsilon$ and W_ε depend smoothly on ε, since $\tilde{Y}_\varepsilon$ is nothing but the vertical average of X_ε.

Let Ψ^ε be the family of symplectomorphisms defined by $\phi^\varepsilon_{\tilde{Y}_\varepsilon + W_\varepsilon} = \phi^\varepsilon_{\tilde{Y}_\varepsilon} \circ \Psi^\varepsilon$ and let Z_ε be its vector field. On the one hand, $\Phi^\varepsilon = \phi^\varepsilon_{\tilde{Y}_\varepsilon}$ is fiber-preserving since $\tilde{Y}_\varepsilon$ is a lift of a vector field on $\mathcal{B}$. On the other hand, on can check in a straightforward way that the vector field X^3_ε of a composition of flows $\phi^\varepsilon_{X^1_\varepsilon} \circ \phi^\varepsilon_{X^2_\varepsilon}$ is given by the formula $X^3_\varepsilon = X^1_\varepsilon + \left(\phi^\varepsilon_{X^1_\varepsilon}\right)_* X^2_\varepsilon$. Therefore, in our case we have $\tilde{Y}_\varepsilon + W_\varepsilon = \tilde{Y}_\varepsilon + \phi^\varepsilon_{\tilde{Y}_\varepsilon}(Z_\varepsilon)$ and thus

$$Z_\varepsilon = \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_*^{-1} (W_\varepsilon).$$

According to Theorem 11, W_ε is Hamiltonian and verifies $\langle W_\varepsilon \rangle = 0$. First, this insures that Z_ε is Hamiltonian. Second, Proposition 15 implies that

$$\langle Z_\varepsilon \rangle = \left(\phi^\varepsilon_{\tilde{Y}_\varepsilon}\right)_*^{-1} \langle W_\varepsilon \rangle = 0$$

since $\tilde{Y}_\varepsilon$ is symplectic and a lift of a vector field on $\mathcal{B}$.

Finally, we show that this decomposition is unique. Indeed, suppose that we have a second decomposition $\phi^\varepsilon_{X_\varepsilon} = \phi^\varepsilon_{\tilde{Y}'_\varepsilon} \circ \phi^\varepsilon_{Z'_\varepsilon}$ with the same properties. The vector field $\tilde{Y}'_\varepsilon$ must be a lift of a vector field on $\mathcal{B}$ since $\phi^\varepsilon_{\tilde{Y}'_\varepsilon}$ is fiber-preserving. On the other hand, as we mentionned before, we have the relation $\tilde{X}_\varepsilon = \tilde{Y}'_\varepsilon + \phi^\varepsilon_{\tilde{Y}'_\varepsilon}\left(Z'_\varepsilon\right)$. Arguing as before, we can show that $\phi^\varepsilon_{\tilde{Y}'_\varepsilon}\left(Z'_\varepsilon\right)$ is a Hamiltonian vector field with vanishing vertical average. Now, Theorem 11 tells us that the decomposition $X_\varepsilon = \tilde{Y}_\varepsilon + W_\varepsilon$ is unique and thus $\tilde{Y}'_\varepsilon = \tilde{Y}_\varepsilon$ and $Z'_\varepsilon = Z_\varepsilon$. ☐

As an application, the following theorem gives a normal form for regular deformations of a given regular CI system.

Theorem 17. *Let* $\left(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B}\right)$ *a regular CI system. If* H_ε *is a regular deformation of* H_0*, then there exist a family of functions* $I_\varepsilon \in \pi^* \left(C^\infty\left(\mathcal{B}\right)\right)$ *and a family of Hamiltonian symplectomorphisms* $\phi^\varepsilon_{X_{G_\varepsilon}}$*, with* $\langle G_\varepsilon \rangle = 0$ *such that*

$$H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$$

for each ε*.*

Proof. By definition, H_ε is a regular deformation of H_0 if there exist a family of functions $J_\varepsilon \in \pi^* \left(C^\infty\left(\mathcal{B}\right)\right)$ and a family of symplectomorphisms ϕ^ε such that $H_\varepsilon = J_\varepsilon \circ \phi^\varepsilon$. On the other hand, Theorem 16 insures that ϕ^ε decomposes into $\phi^\varepsilon = \Phi^\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$, where Φ^ε is fiber-preserving and $\langle G_\varepsilon \rangle = 0$. Therefore, we have $H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$, where the function $I_\varepsilon = J_\varepsilon \circ \Phi^\varepsilon$ is indeed an element of $\pi^* \left(C^\infty\left(\mathcal{B}\right)\right)$ since Φ^ε is fiber-preserving. □

Remark. We can show [15, 14] that the families I_ε and $\phi^\varepsilon_{X_{G_\varepsilon}}$ in the previous theorem are actually unique provided we assume that H_0 is *non-degenerate*. Nondegeneracy conditions are those used in K.A.M. theories, like for example those introduced by Arnol'd [2], Kolmogorov [7], Bryuno [3] or Rüssmann [16]. They are all open conditions on the Hamiltonians in $\pi^* \left(C^\infty\left(\mathcal{B}\right)\right)$. Imposing Rüssmann's condition (the weakest one) on H_0 implies that the fibration in lagrangian tori along which H_0 is constant is unique. This allows to show the uniqueness of the families I_ε and $\phi^\varepsilon_{X_{G_\varepsilon}}$.

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