DIRAC INDEX CLASSES AND THE NONCOMMUTATIVE SPECTRAL FLOW

ERIC LEICHTNAM AND PAOLO PIAZZA

May 7 2001

Abstract. We present a detailed proof of the existence-theorem for *noncommutative* spectral sections, see [32]. We apply this result to various index-theoretic situations, extending to the noncommutative context results of Melrose-Piazza, Dai-Zhang and Booss-Wojciechowski. In particular, we prove a variational formula, in $K_*(C^*_r(\Gamma))$, for the index classes associated to 1-parameter family of Dirac operators on a Γ-covering with boundary; this formula involves a *noncommutative spectral flow* for the boundary family. Next, we establish an additivity result, in $K_*(C^*_r(\Gamma))$, for the index class defined by a Dirac-type operator associated to a closed manifold M and a map $r : M \to B\Gamma$ when we assume that M is the union along a hypersurface F of two manifolds with boundary: $M = M_+ \cup_F M_-$. Finally, we prove a *defect formula* for the signature-index classes of two cut-and-paste equivalent pairs $(M_1, r_1 : M_1 \to B\Gamma)$ and $(M_2, r_2 : M_2 \to B\Gamma)$, with

$$M_1 = M_+ \cup_{(F,\phi_1)} M_- \quad \text{and} \quad M_2 = M_+ \cup_{(F,\phi_2)} M_- .$$

The formula involves the noncommutative spectral flow of a suitable 1-parameter family of twisted signature operators on F. We give applications to the problem of cut-and-paste invariance of Novikov's higher signatures on closed oriented manifolds.

Contents

1. Introduction and main results

The Connes-Moscovici higher index theorem on Galois coverings [8] can be seen as a family-index theorem but with a *noncommutative* parameter space given by the reduced C^*-algebra of the covering group [7]. It is natural to ask to what extent results established for families of Dirac operators parametrized by a topological space X can be carried over to the noncommutative context. The past 20 years of noncommutative differential geometry, a branch of Mathematics founded by Alain Connes, show that such generalizations have, besides their own beauty, deep and interesting geometric applications. See [7] for a wide and fascinating treatment of many of these generalizations.

We consider the following results in family-index theory:

- on a closed manifold N, the existence of spectral sections for a family of Dirac operators with trivial index class ([27] [28]);
- on a manifold with boundary M, the definition of the Atiyah-Patodi-Singer index class and of the b-index class associated to the choice of a spectral section for the boundary family ([27] [28]);
- the equality of these two index classes ([27] [28])
- the relative index theorem, giving a formula for the difference of two index classes associated to two different choices of spectral sections ([27] [28])
- on a closed manifold $N = M_+ \cup_F M_-$, union along a hypersurface F of two manifolds with boundary, the splitting formula for the index class associated to a family of Dirac operators ([9]).
- the notion of higher spectral flow and its relationships with family index theory ([10])

For an informal presentation of some of these results we refer the reader to the surveys [30] [25]. We shall also consider the following result in ordinary (i.e. *numerical*) index theory:

- Let $M_1 = M_+ \cup_{(F,\phi_1)} M_-$ and $M_2 = M_+ \cup_{(F,\phi_2)} M_-$, $\phi_j \in \mathrm{Diff}(F)$, $\partial M_+ = F = -\partial M_-$ be two even-dimensional manifolds that are cut-and-paste equivalent; let $D\!\!\!/_1$, $D\!\!\!/_2$ be two cut-and-paste equivalent operators of Dirac-type. Then the difference $\mathrm{ind} D\!\!\!/_2^+ - \mathrm{ind} D\!\!\!/_1^+$ is equal to the spectral flow of a natural family of operators on F ([5]).

The main goal of this paper is to establish noncommutative generalizations of the above results and give geometric applications to the problem of cut-and-paste invariance of Novikov's higher signatures on a closed oriented manifold.

The first result, i.e. the existence of *noncommutative* spectral sections, was claimed in the odd-dimensional case by F. Wu in the preprint [32]. Unfortunately Wu's preprint was never published; moreover, as pointed out to us by Michel Hilsum, the proof given by Wu contained one unjustified step. For these reasons we felt it was necessary to present a full proof of the existence of noncommutative spectral sections; the missing step turned out to be non-trivial and one section of this paper is entirely devoted to a rigorous proof of this particular result. Notice that there was much more in Wu's preprint than just the existence-theorem for noncommutative spectral sections. Of course we cannot report here on Wu's further results and we strongly feel that it is a pity that the interesting Mathematics established in [32] will most likely never be published.

Some of our papers on higher index theory freely use the existence-theorem stated in Wu's unpublished preprint as well as the consequent definition, given there, of *noncommutative spectral flow* (this definition is an immediate generalization of the family case treated in [10]).

In these articles of ours, [18], [19], [21], [15] (Appendix), *some* of the above generalizations are presented.

A first example is the definition, in the *even-dimensional* case, of the b-index class associated to a Dirac operator on a Galois covering with boundary once a spectral section for the boundary operator is chosen, see [18]. (Notice that our main interest in [18] was in proving an index *formula*, in noncommutative topological de Rham homology, for the Chern character of such an index class.)

Another example, always from [18], is a formula for the Chern character of the noncommutative spectral flow in terms of higher eta invariants.

The additivity formula is already mentioned in [21].

The existence-theorem for spectral sections on even-dimensional closed manifolds is implicitly proved in [19] (assuming the theorem in the odd dimensional case).

Finally, always in [19], we prove the relative index theorem for *even dimensional* Galois coverings with boundary.

In writing this article we were also motivated by the wish of completing the picture that has emerged so far from these articles. We shall be more specific in each particular section.

The paper is organized as follows. In Section 2 we recall the notion of noncommutative spectral section and prove the fundamental result that a noncommutative spectral section for a Dirac-type operator exists if and only if the index class defined by this operator is trivial in K-Theory. We also recall the notion of difference class and of noncommutative spectral flow. In Section 3 we recall from [18] and [33] the definition of b-index class and APS-index class associated to a Dirac-type operator on an *even-dimensional* Galois covering with boundary; these classes depend on the choice of a noncommutative spectral section for the boundary operator. We then prove the equality of this two index classes. We end Section 3 by extending these results to odd dimensions, using suspension as in [28]. In Section 4, following ideas of Bunke and Dai-Zhang, we prove an additivity formula for index classes. In Section 5 we discuss applications to the problem of cut-and-paste invariance for higher signatures. Let us briefly explain what is the problem and what are the techniques that we shall be using.

Recall from [14] that two manifolds M_1, M_2 are SK-equivalent if

$$M_1 = M_+ \cup_{(F,\phi_1)} M_- \,, \quad M_2 = M_+ \cup_{(F,\phi_2)} M_- \tag{1}$$

with $\partial M_+ = F = -\partial M_-$ and $\phi_j \in \mathrm{Diff}(F)$. In words, M_1 and M_2 are obtained by gluing two manifolds with boundary but the gluing diffeomorhisms are different. Incidentally, SK stands for **S***chneiden*=Cutting and **K***leben*=Pasting.
The signature of a manifold is a SK-invariant: $\sigma(M_1) = \sigma(M_2)$ for M_1, M_2 as above. An *analytic* proof of this fact is given in [5]. The argument given there is

a consequence of a more general formula concerning the numerical indeces of two Dirac-type operators obtained one from the other by a cut-and-paste construction; the formula expresses the difference of the numerical indeces in terms of the spectral flow of a suitable 1-parameter family of operators on F. For the particular case of the signature operator this spectral flow turns out to be zero, as it is simply the spectral flow of a 1-parameter family $\{D_F(\theta)\}_{\theta\in S^1}$, of odd signature operators on F parametrized by a path of metrics. In this last step the cohomological significance of the zero-eigenvalue has been used. Since the signature of a manifold is equal to the index of the signature-operator, we obtain finally

$$\sigma(M_2)-\sigma(M_1)=\mathrm{ind}D_2-\mathrm{ind}D_1=\mathrm{sf}(\{D_F(\theta)\}_{\theta\in S^1})=0 \tag{2}$$

In Section 5 we shall be concerned with the higher analogue of this argument.

We thus consider a pair $(M, r: M\to B\Gamma)$, with M a closed oriented manifold and r a continuous map into the classifying space of a dicrete group Γ. To such a pair one can attach a set of numbers, the Novikov's higher signatures $\sigma(M,r;[c])$ defined by

$$\sigma(M,r;[c])=<L(M)\cup r^*[c],[M]>\ ,$$

with $[c]\in H^*(B\Gamma,\mathbb{Q})=H^*(\Gamma,\mathbb{Q})$. Two pairs $(M_1,r_1: M_1\to B\Gamma)$ and $(M_2,r_2: M_2\to B\Gamma)$ are said to be SK-equivalent if M_1 and M_2 satisfy (1) and $(r_1)_{|M_+}\simeq (r_2)_{|M_+}$, $(r_1)_{|M_-}\simeq(r_2)_{|M_-}$, where $\simeq$ means homotopy equivalence. One can ask whether two SK-equivalent pairs (M_1,r_1), (M_2,r_2) have the same higher signatures. It was observed by John Lott in [24] (Remark 4.1), that results in [14], [29] can be reinterpreted as giving examples where the answer to this question is negative; one can then try to find conditions on Γ and F so as to ensure that the higher signatures are indeed SK-invariants. The first result in this direction is given by Leichtnam-Lott-Piazza in [15], as a Corollary of the higher APS-index formula proved there and the higher index formula proved in [22] (see Corollary 0.4 in [15]). The condition on the group Γ is that it be Gromov hyperbolic or virtually nilpotent. The condition on F is recalled in detail in Sect. 5.2/5.3 below: it is an *homotopy-invariant condition* of the pair $(F,r_{|F})$ and can be briefly formulated as a gap condition, *in middle degree*, for the spectrum of the differential-form Laplacian on the Γ-covering associated to $(F,r_{|F})$. This condition was first introduced by Lott in [23], where a regularization of the apriori divergent integral defining the higher eta invariant *for the signature operator* was proposed. These ideas were further pursued in [19], [24], [15].

A different treatment of the cut-and-paste result proved in [15] was subsequently given in [16]. In that paper techniques from algebraic surgery are used; the condition on F remains the same but the group Γ must only satisfy the property that the Baum-Connes map be rationally injective.

In Section 5 we shall reobtain these results using a higher version of the result of Booss-Wojciechowski, i.e. a higher analogue of (2). We briefly describe the result. Let us denote by $\mathcal{D}^{\mathrm{sign},r_j}_{M_j}$, $j=1,2$, the Mishchenko-Fomenko signature operators associated to the two SK-equivalent pairs (M_j,r_j) and a choice of metric g_j on M_j; then for the corresponding index classes, in $K_*(C^*_r(\Gamma))$, the following *defect formula* holds:

$$\mathrm{Ind}\,\mathcal{D}^{\mathrm{sign},r_2}_{M_1}-\mathrm{Ind}\,\mathcal{D}^{\mathrm{sign},r_1}_{M_1}=\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1})\ . \tag{3}$$

On the right hand side of the above formula the *noncommutative* spectral flow of a suitable 1-parameter family of twisted signature operators on F appears. It is important to notice that, in contrast with the numerical case treated in (2), the noncommutative spectral flow appearing in (3) will in general be different from 0, as it is associated to a family of *twisted* signature operators, with the twisting bundle depending on the parameter θ. A similar formula holds for any Dirac-type operator once the notion of SK-equivalent operators is given. By employing *symmetric* spectral sections, as in [19], we shall show that under the cited assumption on F this noncommutative spectral flow is automatically zero. This fact, together with the assumption that the Baum-Connes assembly map is rationally injective, allows one to reobtain the results of [16].

Notice that *in the even dimensional case* a recent preprint [12] of Hilsum sharpens further these results by slightly weakening the assumptions on F. We refer the reader to his paper for the detailed statement of this weaker assumption and also for a somewhat different treatment of boundary value problems in the noncommutative context, more in the spirit of [5]. It would be interesting to know whether the higher APS index formula proved in [18] can be established for these more general boundary value problems (one advantage of spectral sections being that they are specific enough for local index-theory arguments to be carried over).

Notice also that the cut and paste results in the odd-dimensional case are somewhat more involved than in the even-dimensional case and cannot be obtained by simply crossing with S^1. This phenomenon is already present in [15], [16] and will be further clarified in the present paper.

As a final remark we point out that our K-theoretic approach to the cut-and-paste invariance of higher signatures can be extended to foliations: in that case the higher signatures are defined in [2] and one can ask whether these higher signatures are invariant under an appropriate notion of cut and paste. One of the advantages of the proof presented here is that can be reproposed, step by step, in the foliation case. This is not the case either with the original approach in [15] (which uses *local* index formulae) or with the algebraic-surgery approach of [16]. In fact, finding a proof that was easily extendable to the foliation case was one of the motivations for proving formula (3). We shall deal with the technicalities of this program in a future publication.

Acknowledgements. We thank Michel Hilsum for alerting us of the unjustified step in Wu's preprint and Etienne Blanchard for a helpful comment. We heartily thank Ralph Meyer for pointing out the relevance of the notion of very full projection and for suggesting the statement of Theorem 1 as well as the structure of its proof. The second author wishes to thank John Lott for interesting discussions concerning the cut and paste invariance of higher signatures.

Both authors are memebers of the RTN "Geometric Analysis" of the European Community. The second author has received financial support from the "Short-term mobility program" of the *Consiglio Nazionale delle Ricerche* and from *Ministero dell'Università e della Ricerca Scientifica e Tecnologica.* The writing of this paper was completed in April 2001 while the second author was visiting the MSRI (Berkeley) as a member of the *Spectral Invariants* program. Financial support for this visit, from March 1st to May 12th, was provided by MSRI and by the *Istituto Nazionale di Alta Matematica* through the GNSAGA.

2. Proof of the existence of noncommutative spectral sections

We begin this section with two functional analytic results that will be needed in the proof of the existence-theorem. The definition of noncommutative spectral section and the statement of the existence-theorem are given in Section 2.4 below.

2.1. **Very full projections and a lifting theorem.**

We consider in this section a unital C^*-algebra A and denote by $H_A = l^2(A) = l^2(\mathbb{N})\otimes_{\mathbb{C}} A$ the standard $A-$Hilbert module. We consider $\mathcal{B}_A$ the algebra of $A-$linear continuous adjointable operators from H_A to itself and denote by $\mathcal{K}_A$ the subalgebra of $\mathcal{B}_A$ of all the operators T such that both T and T^* are $A-$compact. We then have the following exact sequence:

$$0 \to \mathcal{K}_A \to \mathcal{B}_A \to \mathcal{C}_A = \mathcal{B}_A/\mathcal{K}_A \to 0$$

where $\mathcal{C}_A$ denotes the (generalized) Calkin algebra. We recall that (see [3]) the $K-$theory groups $K_*(\mathcal{B}_A)$ vanish and that $K_*(\mathcal{K}_A) \simeq K_*(A)$. The standard six terms exact sequence in $K-$Theory gives the following isomorphism δ called the index map:

$$\delta : K_0(\mathcal{C}_A) \to K_1(\mathcal{K}_A) \simeq K_1(A) \tag{4}$$

We shall now recall what a *very full* projection is, see [3]. The relevance of this notion in our problem has been pointed out to us by Ralf Meyer.

Definition 1. A projection $p\ (= p^*)$ of a unital C^*-algebra A is a said to be very full if there exists an isometry $u \in A$ $(u^*u = 1 = 1_A)$ such that $uu^* \leq p$. In other words, p contains a projection which is equivalent to 1.

The following Theorem and the precise structure of its proof has been suggested to us by Ralf Meyer.

Theorem 1. *(Ralf Meyer) Let p be a very full projection of $\mathcal{C}_A$ such that the K-Theory class $[p]$ of p in $K_0(\mathcal{C}_A)$ is zero (i.e. the index $\delta(p)$ is zero) and $1-p$ is very full. Then there exists a projection $P \in \mathcal{B}_A$ such that $\pi(P) = p$ where $\pi : \mathcal{B}_A \to \mathcal{C}_A = \mathcal{B}_A/\mathcal{K}_A$ is the canonical projection.*

Proof. The proof of the Theorem relies on the next two Propositions.

Proposition 1. *Let p be a very full projection of $\mathcal{C}_A$. Then there exists an isometry $W \in \mathcal{B}_A$ such that $\pi(W)\pi(W^*) \leq p$.*

Proof of the Proposition. By assumption there exists $W_1 \in \mathcal{B}_A$ such that $\pi(W_1)$ is an isometry of $\mathcal{C}_A$ satisfying $\pi(W_1)\pi(W_1)^* \leq p$ and $W_1^*W_1 - \mathrm{Id} \in \mathcal{K}_A$. Recall that $l^2(A) = \oplus^{\perp}_{n\geq 0} Ae_n$; for any $N \in \mathbb{N}$, we denote by $P_N \in \mathcal{B}_A$ the projection from $l^2(A)$ onto $\mathrm{Im}\, P_N = \oplus^{\perp}_{n\geq N} Ae_n$. Then there exists $N \in \mathbb{N}$ such that: $||P_N(W_1^*W_1 - \mathrm{Id})P_N|| \leq \frac{1}{4}$ where $||\cdot||$ denotes the C^*- norm of $\mathcal{B}_A$.

Now we consider and fix a unitary isomorphism $\chi : l^2(A) \to \mathrm{Im}\, P_N \subset l^2(A)$

We have:

$$||\chi^{-1}P_N(W_1^*W_1 - \mathrm{Id})P_N\chi|| \leq \frac{1}{4}; \tag{5}$$

we identify $P_N\chi$ with an element of $\mathcal{B}_A$ so that $(P_N\chi)^* = \chi^{-1}P_N$. We then set $W_2 = W_1P_N\chi \in \mathcal{B}_A$ so that $W_2^* = \chi^{-1}P_N W_1^*$. From (5) we deduce that:

$$||W_2^*W_2 - \mathrm{Id}\,|| = ||\chi^{-1}P_N(W_1^*W_1 - \mathrm{Id})P_N\chi|| \leq \frac{1}{4}$$

So $W_2^*W_2 \in \mathcal{B}_A$ is invertible and $\operatorname{Im} W_2^* = l^2(A)$. Therefore W_2^* admits a polar decomposition $W_2^* = V\sqrt{W_2W_2^*}$, $\operatorname{Im} W_2 = \operatorname{Im} W_2W_2^*$ is closed in $l^2(A)$, $\ker W_2^* = \ker W_2W_2^*$ and

$$l^2(A) = \operatorname{Im} W_2W_2^* \oplus^{\perp} \ker W_2W_2^* \tag{6}$$

(See Theorem 15.3.8 of [31] for details). Moreover V^*V (resp. VV^*) is the orthogonal projection onto $\operatorname{Im} W_2W_2^*$ (resp. $\operatorname{Im} W_2^*W_2 = l^2(A)$), so $VV^* = \operatorname{Id}$. From the definition of W_2 we get $W_2W_2^* = W_1W_1^* + W_1(P_N - \operatorname{Id})W_1^*$ and thus $\pi(W_2W_2^*) = \pi(W_1W_1^*)$. Since by hypothesis $\pi(W_2W_2^*) = \pi(W_1W_1^*) \leq p$, we obtain the Proposition as a consequence of the following Lemma by taking $W = V^*$.

Lemma 1. *We have $\pi(V^*V) = \pi(W_2W_2^*)$.*

Proof. Using the decomposition (6) and the fact that $\operatorname{Im} W_2W_2^*$ is closed and defines a $A-$Hilbert module, one sees that $W_2W_2^*$ induces a (bi-)continuous bijection:

$$W_2W_2^* : \operatorname{Im} W_2W_2^* \to \operatorname{Im} W_2W_2^*$$

still denoted $W_2W_2^*$. We denote by S its inverse; by extending S by 0 on $\ker W_2W_2^*$, we get an element of $\mathcal{B}_A$ still denoted S. Thus, by definition of the polar decomposition $W_2^* = V\sqrt{W_2W_2^*}$, we have $W_2W_2^*S = V^*V$. Now we consider the universal representation of the Calkin algebra $j : \mathcal{C}_A \to B(H_{\mathcal{C}_A})$. By applying $j \circ \pi$ to the previous identity we get: $j(\pi(W_2W_2^*))j(\pi(S)) = j(\pi(V^*V))$; so the range of the projection $j(\pi(V^*V))$ is contained in the range of the projection $j(\pi(W_2W_2^*))$. Moreover from the identity $W_2W_2^* = V^*VW_2W_2^* = W_2W_2^*V^*V$, we get that the range of $j(\pi(W_2W_2^*))$ is contained in the one of $j(\pi(V^*V))$. Since j is injective, we see that $\pi(V^*V) = \pi(W_2W_2^*)$ which proves the Lemma.

Proposition 2. *Let p be a very full projection of $\mathcal{C}_A$ such that $1-p$ is also very full. Then there exists a projection p'' of $\mathcal{C}_A$ such that for any $k, l \in \mathbb{N}^*$, one can find an $A-$linear bicontinuous isomorphism:*

$$\chi_{k,l} : l^2(A) \to (\,\Pi_{i=1}^k l^2(A)\,) \oplus l^2(A) \oplus (\,\Pi_{j=1}^l l^2(A)\,) = \mathcal{H}_{k,l}$$

such that

$$p = \chi_{k,l}^{-1}(1_k \oplus p'' \oplus 0_l)\chi_{k,l} \tag{7}$$

where $1_k \oplus p'' \oplus 0_l$ denotes the obvious diagonal-by-blocks element of the Calkin algebra $\mathcal{C}_A(k,1,l)$ associated with $\mathcal{H}_{k,l}$ and $u \to \chi_{k,l}^{-1}u\chi_{k,l}$ induces an isomorphism between $\mathcal{C}_A$ and $\mathcal{C}_A(k,1,l)$.

Proof. We shall treat (only) the case $k = l = 1$ from which the general case follows immediately.

By considering a decomposition $l^2(A) \simeq l^2(A) \oplus l^2(A)$, we choose and fix a projection $\Pi_1 \in \mathcal{B}_A$ such that $\ker \Pi_1$ and $\operatorname{Im} \Pi_1$ are both isomorphic to $l^2(A)$. According to Proposition 1, we fix an isometry $W \in \mathcal{B}_A$ such that $\pi(W)\pi(W^*) \leq p$. We set $\widehat{W} = W\Pi_1$, so $\widehat{W}^* = \Pi_1 W^*$. Since $W^*W = \operatorname{Id}$ we get:

$$\widehat{W}^*\widehat{W} = \Pi_1,\ \widehat{W}\widehat{W}^* = W\Pi_1W^*,\ WW^*\widehat{W}\widehat{W}^* = \widehat{W}\widehat{W}^*WW^* = \widehat{W}\widehat{W}^*$$

From this, we deduce that $\widehat{W}\widehat{W}^* \leq WW^*$. So $WW^* - \widehat{W}\widehat{W}^* = W(\operatorname{Id} - \Pi_1)W^*$ is a projection,

$$\operatorname{Im} WW^* = \operatorname{Im} \widehat{W}\widehat{W}^* \oplus^{\perp} \operatorname{Im}(WW^* - \widehat{W}\widehat{W}^*)$$

and we get:

$$l^2(A) = \operatorname{Im} \widehat{W}\widehat{W}^* \oplus^{\perp} \operatorname{Im}(WW^* - \widehat{W}\widehat{W}^*) \oplus^{\perp} \ker WW^* \tag{8}$$

Since the map:

$$\operatorname{Im} \widehat{W}^*\widehat{W} = \operatorname{Im} \Pi_1 \simeq l^2(A) \to \operatorname{Im} \widehat{W}\widehat{W}^*$$
$$y \to \widehat{W}(y)$$

induces an isomorphism, one sees that $\operatorname{Im} \widehat{W}\widehat{W}^*$ is isomorphic to $l^2(A)$. Moreover, since $WW^* - \widehat{W}^*\widehat{W} = W(\operatorname{Id} - \Pi_1)(W(\operatorname{Id} - \Pi_1))^*$, one proves in the same way that $\operatorname{Im}(WW^* - \widehat{W}^*\widehat{W}) \simeq l^2(A)$. Thanks to Kasparov's Stabilization Theorem, we then have:

$$\operatorname{Im}(WW^* - \widehat{W}^*\widehat{W}) \oplus^{\perp} \ker WW^* \simeq l^2(A). \tag{9}$$

Now we go back to the proof of the Proposition. Since we have $p\pi(WW^*) = \pi(WW^*)p = \pi(WW^*)$ (ie $p \geq \pi(WW^*)$) and $\pi(WW^*) \geq \pi(\widehat{W}\widehat{W}^*)$, we see that $p \geq \pi(\widehat{W}\widehat{W}^*)$ and $p \geq \pi((WW^* - \widehat{W}\widehat{W}^*)$. Using the decomposition (8), we can identify $\mathcal{B}_A$ as a set of $(3,3)-$ matrices, accordingly, we then see that $p (\in \mathcal{C}_A)$ is of the following diagonal form by blocks:

$$p = 1 \oplus 1 \oplus q_1 \tag{10}$$

We set $1 \oplus q_1 = p'$, p' is a projection of the Calkin algebra associated with the left hand-side of (9). Since $1-p$ is very full, $1-p'$ is also very full. Applying (10) with $1-p'$ in place of p, we obtain a decomposition:

$$\operatorname{Im}(WW^* - \widehat{W}^*\widehat{W}) \oplus^{\perp} \ker WW^* \simeq l^2(A) \oplus l^2(A) \tag{11}$$

such that $1-p' = q_2 \oplus 1$ (diagonal by blocks) where q_2 is a projection in $\mathcal{C}_A$. So, from (8) and (11), we get a decomposition $l^2(A) \simeq l^2(A) \oplus l^2(A) \oplus l^2(A)$ such that:

$$p = 1 \oplus (1-q_2) \oplus 0.$$

One then gets the Proposition by setting $p'' = 1 - q_2$.

End of the proof of Theorem 1.

We use the projection $p'' \in \mathcal{C}_A$ associated to p as in Proposition 2.

Lemma 2. *The $K-$theory class* $[1]$ *of* 1 *in* $K_0(\mathcal{C}_A)$ *is zero.*

Proof of the Lemma. We recall that the index map (1) δ is an isomorphism and is also the connecting map associated with the six terms exact sequence in $K-$Theory. Since the projection 1 in the Calkin algebra $\mathcal{C}_A$ is the image of the projection defined by Id in $\mathcal{B}_A$, we have $\delta([1]) = 0$. Since δ is an isomorphism, the Lemma is proved.

Since $[p]$ is (by assumption) zero, the previous Lemma then shows that $[p''] = 0$. Thus there exists $k, l' \in \mathbb{N}^*$ and a unitary element $u \in U(M_{k+1+l'}(\mathcal{C}_A))$ such that:

$$u(1_k \oplus 0_1 \oplus 0_{l'})u^{-1} = 1_k \oplus p'' \oplus 0_{l'}.$$

We set $l = k + 2l' + 1$ and $z = \operatorname{diag}(u, u^{-1})$. We then have:

$$z(1_k \oplus 0_1 \oplus 0_l)z^{-1} = 1_k \oplus p'' \oplus 0_l.$$

Since z belongs to the connected component of 1 in $U(M_{k+1+l}(\mathcal{C}_A))$ (see [3]) , we can find Z in $U(M_{k+1+l}(\mathcal{B}_A))$ such that $\pi(Z) = z$. Now we apply Proposition 2 with k and $l = k + 2l' + 1$ and we set:

$$P = \chi_{k,l}^{-1} Z(\operatorname{Id}_k \oplus 0_1 \oplus 0_l)Z^{-1}\chi_{k,l} \in \mathcal{B}_A. \tag{12}$$

We thus get a projection P of $\mathcal{B}_A$, using the two equalities (7), (12) and the fact that $\pi(Z) = z$, one sees that that $\pi(P) = p$ which proves Theorem 1.

2.2. **Spectral cuts and very full projections.**

We now consider a finitely generated group Γ ; we shall apply the above results in a particular case to be described below, where the C^*-algebra A will be equal to $\Lambda = C_r^*(\Gamma)$, the reduced C^*-algebra associated to the Γ. We also consider a smooth compact riemannian manifold M of dimension $2m+1$, a continuous map $f : M \to B\Gamma$ and a complex hermitian clifford module $E \to M$ endowed with a unitary clifford connection. Let $\widetilde{M} \to M$ be the $\Gamma-$normal cover of M associated with $f : M \to B\Gamma$. Let $\mathcal{V}_f = \Lambda \times_\Gamma \widetilde{M} \to M$ be the $\Lambda-$flat bundle associated to these data. Then we denote by $\mathcal{D}$ the $\Lambda-$Dirac type operator acting on $C^\infty(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$, the set of smooth sections of the bundle $E \otimes_{\mathbb{C}} \mathcal{V}_f \to M$. In fact $\mathcal{D}$ defines a self-adjoint unbounded regular operator acting on the $\Lambda-$Hilbert module $L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$, the completion of $C^\infty(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ with respect to the $\Lambda-$hermitian scalar product associated to the above data. Of course, $L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ is isomorphic to $l^2(\Lambda)$, and the algebra of continuous $\Lambda-$linear adjointable operators $BL^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ is isomorphic to $\mathcal{B}_\Lambda$. We recall that for any $F \in C^0([-\infty, +\infty], \mathbb{C})$, $F(\mathcal{D})$ is well defined as an element of $BL^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$. Moreover if $KL^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ denotes the ideal of adjointable $\Lambda-$compact operators then the following Calkin algebra:

$$C_\Lambda(E) = \frac{BL^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)}{KL^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)}$$

is isomorphic to $\mathcal{C}_\Lambda$.

Now we recall the following:

Definition 2. A smooth map $\chi \in C^\infty(\mathbb{R}, [0,1])$ is said to be a spectral cut if there exists two reals $a < b$ such that for any $t < a$, $\chi(t) = 0$ and for any $t > b$, $\chi(t) = 1$.

Theorem 2. *Let χ be any spectral cut,then $\chi(\mathcal{D})$ and $\mathrm{Id} - \chi(\mathcal{D})$ define two very full projections of the Calkin algebra $C_\Lambda(E)$.*

Proof. We will treat only the case of $\chi(\mathcal{D})$, the case of $\mathrm{Id} - \chi(\mathcal{D})$ being similar. We shall need the following

Technical Lemma. *Let χ be a spectral cut. There exists an orthonormal sequence $(e_k)_{k \in \mathbb{N}}$ of elements of $L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that for any $k, l \in \mathbb{N}$ $< e_k; e_l >= \delta_{k,l} 1_\Gamma$ and $\chi(\mathcal{D})(e_k) = e_k$.*

Granting this result, we shall now prove Theorem 2. We denote by $F \subset L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ the closure of $\oplus^\perp_{k \in \mathbb{N}} \Lambda e_k$. Thus, F is $\Lambda-$Hilbert module contained in the range of $\chi(\mathcal{D})$. We denote by $p_F = \sum_{k \in \mathbb{N}} < \cdot\, ; e_k > e_k$ the orthogonal projection onto F, p_F belongs to $B_\Lambda L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$. We fix a $\Lambda-$unitary isomorphism $\theta : L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f) \to F$ and denote by j the canonical injection $j : F \to L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$. Then $j \circ \theta$ defines an element of $B_\Lambda L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ whose adjoint is given by $(j \circ \theta)^* = \theta^* \circ p_F$. Then one checks easily that:

$$\theta^* \circ p_F \circ j \circ \theta = \mathrm{Id}_{L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)}, \quad j \circ \theta \circ \theta^* \circ p_F = p_F.$$

But, according to the technical Lemma we have

$$\chi(\mathcal{D}) \circ p_F = p_F = (\chi(\mathcal{D}) \circ p_F)^* = p_F \circ \chi(\mathcal{D}),$$

therefore we deduce that, in the Calkin algebra, the projection $\chi(\mathcal{D})$ contains the very full projection p_F. Theorem 2 is proved.

2.3. **Proof of the technical Lemma.**

This section can be skipped at first reading. Recall that the principal symbol of D (acting on sections of E) is $\sigma(D)(x;\xi)=\sqrt{-1}c(\xi)$ where $c(\xi)$ denotes the Clifford multiplication by $\xi\in T_x^*M$; thus the principal symbol of $\mathcal{D}$ is $\sigma(\mathcal{D})(x;\xi)=\sqrt{-1}c(\xi)\otimes_{\mathbb{C}}\mathrm{Id}_{\mathcal{V}_f}$. Let $x\in M$ and $\xi\in S_x^*M$ (the unit cotangent sphere) then $(\sqrt{-1}c(\xi))^2=\mathrm{Id}$ and:

$$E_x=\ker(\sqrt{-1}c(\xi)-\mathrm{Id})\oplus^{\perp}\ker(\sqrt{-1}c(\xi)+\mathrm{Id}). \tag{13}$$

In other words, the two bundles $\ker(\sqrt{-1}c(\xi)\pm\mathrm{Id})\to S_x^*M$ define a direct orthogonal sum decomposition of the trivial pulled back bundle $\pi^*(E_x)\to S_x^*M$ where $\pi: S_x^*M\to\{x\}$ denotes the canonical projection. We start with an elementary result.

Lemma 3. *Let $\xi\in S_x^*M$, we then have:*
1]. For any spectral cut χ, the principal symbol $\sigma(\chi(D))(x;\xi)$ is the orthogonal projection of $\pi^*(E_x)$ onto $\ker(\sqrt{-1}c(\xi)-\mathrm{Id})$.
2].

$$\dim\ker(\sqrt{-1}c(\xi)-\mathrm{Id})=\dim\ker(\sqrt{-1}c(\xi)+\mathrm{Id})=$$
$$\frac{1}{2}\dim E_x\geq 2^{m-1}.$$

Proof. 1]. We denote by $|D|^{-1}$ the operator defined as the inverse of D on $\mathrm{Im}D\in L^2(M,E)$ and by zero on $\ker D$. The identity (13) shows that the principal symbol $\sigma(\mathrm{Id}+\frac{D}{|D|})(x,\xi)$ is the orthogonal projection of $\pi^*(E_x)$ onto $\ker(\sqrt{-1}c(\xi)-\mathrm{Id})$. Since $\chi(D)-(\mathrm{Id}+\frac{D}{|D|})$ is a smoothing operator, we see that $\chi(D)$ and $\mathrm{Id}+\frac{D}{|D|}$ have the same principal symbol, *1]* is thus proved.

2]. This result is implicitly proved in the literature; we give the proof for the sake of completness. By writing the clifford module E_x as a direct sum of irreducible clifford modules, one can reduce oneself to the case where E_x itself is irreducible. Recall then that the Clifford algebra $Cl(\mathbb{C}^{2m+1})$ is isomorphic to the algebra $M_{2^m}(\mathbb{C})\oplus M_{2^m}(\mathbb{C})$ so that $Cl(\mathbb{C}^{2m+1})$ admits to inequivalent complex irreducible modules $\mathcal{S}^+,\mathcal{S}^-$ of the same dimension 2^m, thus E_x is one of the $\mathcal{S}^\pm$ and $\dim E_x=2^m$. Moreover, there exists an algebra isomorphism θ:

$$\theta: Cl(\mathbb{C}^{2m+2})\to\mathrm{End}_{\mathbb{C}}(\mathcal{S}^+\oplus\mathcal{S}^-).$$

As usual, the $Cl(\mathbb{C}^{2m+1})$ is identified to the even part $Cl^0(\mathbb{C}^{2m+2})$ of the clifford algebra $Cl(\mathbb{C}^{2m+2})$. by the isomorphism induced by the map $\xi\in\mathbb{C}^{2m+1}\to e_{2m+2}\cdot\xi$ where e_{2m+2} is the last vector of the canonical basis of $\mathbb{R}^{2m+2}$. For any $\xi\in\mathbb{C}^{2m+1}$, we set $c(\xi)=\theta(e_{2m+2}\cdot\xi)$, the operator $c(\xi)$ sends $\mathcal{S}^\pm$ into itself and its restriction to E_x (identified to one of the $\mathcal{S}^\pm$) coincides with the clifford multiplication (also denoted $c(\xi)$ above) induced by the principal symbol $\sigma(\mathcal{D})(x;\xi)$ of $\mathcal{D}$. So we have to prove that for any $\xi\in S^{2m}=\{\eta\in\mathbb{R}^{2m+1}\,/\,||\eta||=1\}$:

$$\dim\ker(\sqrt{-1}c(\xi)-\mathrm{Id})_{|\mathcal{S}^\pm}=\dim\ker(\sqrt{-1}c(\xi)+\mathrm{Id})_{|\mathcal{S}^\pm}.$$

Since $m\geq 1$ (and thus $2m+2>2$), Definition 3.20 and Proposition 3.21 of [4] show that:

$$\mathrm{Tr}\sqrt{-1}c(\xi)_{|\mathcal{S}^+}=\mathrm{Tr}\sqrt{-1}c(\xi)_{|\mathcal{S}^-} \tag{14}$$

Now we consider the Chirality operator $\beta = (\sqrt{-1}^{m+1})e_1 \dots e_{2m+1}$; in this formula $(e_1, \dots, e_{2m+1})$ is the canonical oriented basis of $\mathbb{R}^{2m+1}$. Let $\xi \in S^{2m}$, recall (see [4] page 109-110) that $\beta \cdot e_{2m+2} \cdot \xi = -e_{2m+2} \cdot \xi \cdot \beta$ and $\theta(\beta)c(\xi) = -c(\xi)\theta(\beta)$.

Since $\theta(\beta)$ sends $\mathcal{S}^{\pm}$ onto $\mathcal{S}^{\mp}$ we then get:

$$\ker(\sqrt{-1}c(\xi) \pm \mathrm{Id})_{|\mathcal{S}^{\pm}} \simeq \ker(\sqrt{-1}c(\xi) \mp \mathrm{Id})_{|\mathcal{S}^{\mp}}. \tag{15}$$

Moreover, since $(\sqrt{-1}c(\xi))^2 = \mathrm{Id}$ we have

$$\mathrm{Tr}\sqrt{-1}c(\xi)_{|\mathcal{S}^{\pm}} = \dim\, \ker(\sqrt{-1}c(\xi) - \mathrm{Id})_{|\mathcal{S}^{\pm}} - \dim\, \ker(\sqrt{-1}c(\xi) + \mathrm{Id})_{|\mathcal{S}^{\pm}} \tag{16}$$

Using the three equations (14), (15) and (16) we get

$$\dim\, \ker(\sqrt{-1}c(\xi) - \mathrm{Id})_{|\mathcal{S}^{\pm}} \;=\; \dim\, \ker(\sqrt{-1}c(\xi) + \mathrm{Id})_{|\mathcal{S}^{\pm}}.$$

which proves the Lemma.

Lemma 4. *Let χ be a spectral cut. There exists a strictly increasing sequence of real numbers $(a_k)_{k\in\mathbb{N}}$ such that χ is equal to 1 on $]a_0 - 1, +\infty[$, $\lim a_k = +\infty$ and for any $k \in \mathbb{N}$, one can find $\phi_k \in C^\infty_{\mathrm{comp}}(]a_k, a_{k+1}[, \mathbb{R})$ and $u_k \in L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that $||\phi_k(\mathcal{D})(u_k)||$ is invertible in Λ.*

Proof. We fix a spectral cut χ_1 whose support is contained in $[a_0 + 1, +\infty[$, thus χ is equal to 1 on an open neighborhood of the support of χ_1. We are going to show that one can find $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$ and $v \in L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that $||\psi(\mathcal{D})\chi_1(\mathcal{D})(v)||$ is invertible in Λ. This will imply the Lemma in the following way: we fix $a_1 > a_0$ such that the support of $\phi_0 = \psi\chi_1$ is contained in $]a_0, a_1[$ then, to construct ϕ_1, we proceed as above, replacing a_0 by a_1. Thus inductively, we construct the a_k and the ϕ_k satisfying the requirements of *1].*

Let U be a (small) nonempty geodesically convex open subset of M over which the bundles E and $\mathcal{V}_f$ are trivial. Let D denote the Dirac operator acting on the sections of E. We fix $x_0 \in U$ and $\xi_0 \in S^*_{x_0}M$, Lemma 3 2] allows us to fix also a non zero vector v_0 in $\ker(\sqrt{-1}c(\xi) - \mathrm{Id})_{|E_{x_0}}$. We consider also a function $b \in C^\infty_{\mathrm{comp}}(U, \mathbb{R})$ which is equal to one near x_0; since $E_{|U} \sim U \times E_{x_0}$ we may define an element of $L^2_{\mathrm{comp}}(U, E)$ by the following formula:

$$u = b(\mathrm{Id} + \Delta)^{-50n-1}(\delta_{x_0})\, v_0 \in L^2_{\mathrm{comp}}(U, E)$$

where Δ denotes the standard Laplace-Beltrami operator acting on functions and δ_{x_0} denotes the Dirac delta function.

Subemma 1. *The point $(x_0, \xi_0, v_0) \in S^*M \times (E_{x_0} \setminus \{0\})$ belongs to the polarization set (see* [11]*) of u.*

Proof of the sublemma. Obviously, (x_0, ξ_0) belongs to the wave front set of u, so there exits $w_0 \in E_{x_0} \setminus \{0\}$ such that (x_0, ξ_0, w_0) belongs to the polarization set of u. Now let w be any vector of $E_{x_0} \setminus \mathbb{C}v_0$, we then consider a linear form $\mathcal{L} \in E^*_{x_0}$ such that $\mathcal{L}(v_0) = 0$ and $\mathcal{L}(w) \neq 0$. Obviously we have:

$$\sqrt{\mathrm{Id} + \Delta}\, \mathcal{L}(u) \equiv 0$$

where $\sqrt{\mathrm{Id} + \Delta}\, \mathcal{L}$ is identified with a $1 \times \dim E_{x_0}$ system of scalar pseudo-differential operators of order one so that $\sigma(\sqrt{\mathrm{Id} + \Delta}\, \mathcal{L})(x_0; \xi_0).w \neq 0$. Then, (by definition) we deduce that (x_0, ξ_0, w) does not belong to the polarization set of u. Thus (x_0, ξ_0, v_0) belongs to the polarization set of u which proves the subemma.

Now, if $\chi_1(D)u$ were identically zero then, fix any $1 \times \dim E_{x_0}$ system of scalar pseudo-differential operators of order one A such that $\sigma(A)(x_0;\xi_0).v_0 \neq 0$. Since Lemma 3 1] shows that $\sigma(\chi_1(D))(x_0;\xi_0).v_0 = v_0$ then $\sigma(A \circ \chi_1(D))(x_0;\xi_0).v_0$ is also not zero. For such an A we would have $A \circ \chi_1(D)(u) \equiv 0$ then , by definition of the polarization set, this would imply that that (x_0, ξ_0, v_0) doesnot belong to the polarization set of u, which contradicts Lemma 1. So $\chi_1(D)u$ is not identically zero.

Now we fix a be a function in $C^{\infty}_{\text{comp}}(U, \mathbb{R})$ which is equal to 1 on an open neibourghood of $\operatorname{supp} u$. Since both E and $\mathcal{V}_f$ are trivial over U, $u.1_\Gamma$ defines an element of $L^2_{\text{comp}}(U, E \otimes_{\mathbb{C}} \mathcal{V}_f)$. Then, one sees immediately that:

$$< \chi_1(\mathcal{D})(u.1_\Gamma); \chi_1(\mathcal{D})(u.1_\Gamma) > = < \chi_1^2(\mathcal{D})(u.1_\Gamma); u.1_\Gamma > = < a\chi_1^2(\mathcal{D})(u.1_\Gamma); u.1_\Gamma > .$$

But since a has compact support in U, it is clear that $a\chi_1^2(\mathcal{D})(u.1_\Gamma) = a\chi_1^2(D)(u).1_\Gamma$. Therefore,

$$< \chi_1(\mathcal{D})(u.1_\Gamma); \chi_1(\mathcal{D})(u.1_\Gamma) > = < a\chi_1^2(D)(u); u > .1_\Gamma = < \chi_1(D)(u); \chi_1(D)(u) > .1_\Gamma$$

is invertible in Λ. Moreover, one checks easily that there exists $\psi \in C^{\infty}_{\text{comp}}(\mathbb{R}, \mathbb{R})$ such that $\psi(D)\chi_1(D)(u) \neq 0$ and thus

$$||\psi(\mathcal{D})\chi_1(\mathcal{D})(u.1_\Gamma)|| = ||\psi(D)\chi_1(D)(u)||_{L^2(M)}.1_\Gamma$$

is invertible in Λ. This proves Lemma 4.

Recall that our goal in this section is to provide a proof of the following technical Lemma:

Let χ be a spectral cut. There exists an orthonormal sequence $(e_k)_{k\in\mathbb{N}}$ of elements of $L^2_\Lambda(M, E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that for any $k, l \in \mathbb{N}$ $< e_k; e_l > = \delta_{k,l} 1_\Gamma$ and $\chi(\mathcal{D})(e_k) = e_k$.

However, by setting for each $k \in \mathbb{N}$:

$$e_k = ||\phi_k(\mathcal{D})(u_k)||^{-1}\phi_k(\mathcal{D})(u_k)$$

one deduces immediately this result from Lemma 4.

2.4. **Existence of spectral sections in the odd dimensional case.**

We begin by recalling the definition of noncommutative spectral section. This definition was given in the unpublished paper [32] and recalled in [33], which is a published paper.

Definition 3. (F. Wu) A projection $\mathcal{P} \in BL^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)$ is said to be a noncommutative spectral section for $\mathcal{D}$ if there exist two spectral cuts χ_1, χ_2 such that $\chi_2 \equiv 1$ on a neighborhood of the support of χ_1, and $\operatorname{Im} \chi_1(\mathcal{D}) \subset \operatorname{Im} \mathcal{P} \subset \operatorname{Im} \chi_2(\mathcal{D})$.

The following Theorem is stated by Wu in [32] [33].

Theorem 3. *There exists a spectral section for $\mathcal{D}$ if and only if the index $\delta([\chi(\mathcal{D})])$ (see (1)) vanishes in $K_1(\Lambda)$.*

Proof. We start by proving that if a spectral section $\mathcal{P}$ exists, then $\delta([\chi(\mathcal{D})] = 0$. Let us fix a spectral cut χ. By the very definition of spectral section, $\mathcal{P}$ and $\chi(\mathcal{D})$ induce the same element in the Calkin algebra $C_\Lambda(E)$. So the $K-$theory class $[\chi(\mathcal{D})] \in K_0(C_\Lambda(E))$ is the image of $[P] \in K_0(B_\Lambda L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f))$ by the map induced by the projection π $B_\Lambda L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f) \to C_\Lambda(E)$. Since $K_0(B_\Lambda L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)) = \{0\}$, we see immediately that $[\chi(\mathcal{D})] = 0$ and thus that $\delta([\chi(\mathcal{D})] = 0$.

We shall now prove the converse. Some of the arguments are already in [32] (but we shall recall them so as to give a complete proof).

First we have the the following Lemma which is a noncommutative generalization of Lemma 1 of [27].

Lemma 5. *(F. Wu) Let $\mathcal{E}$ be a $\Lambda-$Hilbert module isomorphic to $l^2(\Lambda)$. Let $Q = Q^2 = Q^* \in \mathcal{B}_\Lambda(\mathcal{E})$ be a projection and $J \in \mathcal{B}_\Lambda(\mathcal{E})$ such that $||Q-J|| \leq \frac{1}{2}$. Then there is a function f, holomorphic in a neighborhood of the spectrum of $|QJ^*|$, such that $P = f(JQJ^*) \in \mathcal{B}_\Lambda(\mathcal{E})$ is the orthogonal projection onto $JQ(\mathcal{E})$. Furthermore, if $Q - J \in \mathcal{K}_\Lambda(\mathcal{E})$, then $P - Q \in \mathcal{K}_\Lambda(\mathcal{E})$ too.*

Proof. For any $x \in Q(\mathcal{E})$ one has $x = Q(x) = J(x) + (Q-J)(x)$. Hence we have:

$$\forall x \in Q(\mathcal{E}), \quad ||J(x) \geq \frac{1}{2}||x||.$$

From this inequality, one deduces easily that JQ has closed range. Hence $(JQ)^* = QJ^*$ has also closed range and Theorem 15.3.8 of [31] shows that:

$$JQ(\mathcal{E}) = \ker(QJ^*)^\perp = \ker(|QJ^*|)^\perp = |QJ^*|(\mathcal{E})$$

$$\mathcal{E} = \ker(|QJ^*|) \oplus JQ(\mathcal{E}) = \ker(|QJ^*|^2) \oplus JQ(\mathcal{E}) = \ker(JQJ^*) \oplus JQ(\mathcal{E}).$$

Therefore 0 is an isolated point in the spectrum of $|QJ^*|$ and hence of $|QJ^*|^2 = JQJ^*$. Let $f(\lambda) = 1$ for λ in a neighborhood of $Spec(|QJ^*|^2) \setminus \{0\}$ and $f(\lambda) = 0$ for λ near 0. Then f is holomorphic in a neighborhood of $Spec(|QJ^*|^2)$ and:

$$P = f(|QJ^*|^2) = f(JQJ^*)$$

is the orthogonal projection onto $JQ(\mathcal{E})$.

Now we show that if in addition $Q - J \in \mathcal{K}_\Lambda(\mathcal{E})$ then $P - Q \in \mathcal{K}_\Lambda(\mathcal{E})$. To this end, we first observe that, since $P(\mathcal{E}) = JQ(\mathcal{E})$, we have $P(JQ) = JQ$. Since $\pi(JQ) = \pi(Q)$ in the Calkin algebra $\mathcal{C}_\Lambda(\mathcal{E})$, this implies that:

$$\pi(P)\pi(Q) = \pi(P)\pi(JQ) = \pi(PJQ) = \pi(JQ) = \pi(Q) \in \mathcal{C}_\Lambda(\mathcal{E}).$$

Taking adjoints on both sides, noticing that both $\pi(P)$ and $\pi(Q)$ are self-adjoint elements in $\mathcal{C}_\Lambda(\mathcal{E})$, we get:

$$\pi(Q) = \pi(Q)\pi(P). \tag{17}$$

On the otherhand, since $J : Q(\mathcal{E}) \to P(\mathcal{E})$ is an isomorphism, $J^{-1}P \in \mathcal{B}_\Lambda(\mathcal{E})$ is well-defined, satisfying $J(J^{-1}P) = P$. Passing to $\mathcal{C}_\Lambda(\mathcal{E})$ and noticing that $\pi(Q)\pi(J) = \pi(J)\pi(Q)$, we obtain $\pi(Q)\pi(P) = \pi(P)$. From equation (17) we get $\pi(P) = \pi(Q)$ in $\mathcal{C}_\Lambda(\mathcal{E})$ which proves the Lemma.

End of the proof of Theorem (3).

Let us fix a spectral cut χ. We know from Theorems 1 and 2 that there exists a projection $\widetilde{P}$ in $B_\Lambda L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that $\widetilde{P} - \chi(\mathcal{D})$ is a $\Lambda-$compact operator.

As pointed out to us by Michel Hilsum, the existence of such a projection $\widetilde{\mathcal{P}}$ was claimed without explanation in [32]. The end of the proof is now based on the techiques in [27] but with the extra care needed in order to deal with Hilbert modules. The details appeared in [32] and we present them here for the sake of completness.

Let $\varphi \in C^\infty_c(\mathbb{R}; [0,1])$ such that $\varphi(t) = 1$ for $t \in [-1,1]$ and $\varphi(t) = 0$ for $|t| \geq 2$. We set $\varphi_N(t) = \varphi(t/N)$ for $N \in \mathbb{N}^*$. One checks easily that as $N \to +\infty$:

$$\varphi_N(\mathcal{D})(\chi(\mathcal{D}) - Q) \to \chi(\mathcal{D}) - Q$$

in the norm of $\mathcal{K}_\Lambda(\mathcal{E})$. Thus there is N_0 such that:

$$||(\mathrm{Id} - Q) - J|| \leq \frac{1}{2},$$

where

$$J = \mathrm{Id} - \chi(\mathcal{D}) - \varphi_{N_0}(\mathcal{D})(\chi(\mathcal{D}) - Q).$$

Applying Lemma 5, there is a projection $P_1 = P_1^* = P_1^2$ in $\mathcal{B}_\Lambda(\mathcal{E})$ such that $P_1 - Q \in \mathcal{K}_\Lambda(\mathcal{E})$ and

$$(\mathrm{Id} - P_1)(\mathcal{E}) = J(\mathrm{Id} - Q)(\mathcal{E}).$$

Choose another spectral cut χ_0 such that

$$\chi_0(t) = 0 \,\text{on}\, \mathrm{Supp}(1-\chi) \cup [-N_0, N_0].$$

Then we have $(1-\chi_0)(1-\chi) = (1-\chi)$ and $(1-\chi_0)\varphi_{N_0} = \varphi_{N_0}$. Thus we have:

$$(\mathrm{Id} - \chi_0(\mathcal{D}))(\mathrm{Id} - \chi(\mathcal{D})) = (\mathrm{Id} - \chi(\mathcal{D}))\ (\mathrm{Id} - \chi_0(\mathcal{D}))\varphi_{N_0}(\mathcal{D}) = \varphi_{N_0}(\mathcal{D}).$$

This implies that $(\mathrm{Id} - \chi_0(\mathcal{D}))J = J$. Hence we have:

$$(\mathrm{Id} - P_1)(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)) = \mathrm{Im}(\mathrm{Id} - \chi_0(\mathcal{D}))J(\mathrm{Id} - Q) \subset (\mathrm{Id} - \chi_0(\mathcal{D}))(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)).$$

Let χ_1 be a spectral cut such that $\chi_0(t) = 1$ on a neighborhood of $\mathrm{Supp}(\chi_1)$. Then we get, for any $x \in L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)$:

$$\chi_1(\mathcal{D})(P_1(x)) = \chi_1(\mathcal{D})(x) - \chi_1(\mathcal{D})((\mathrm{Id} - P_1)(x)).$$

Since $\mathrm{Im}(\mathrm{Id} - P_1) \subset (\mathrm{Id} - \chi_0(\mathcal{D}))(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f))$, there is $y \in L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that $(\mathrm{Id} - P_1)(x) = (\mathrm{Id} - \chi_0(\mathcal{D}))(y)$. Hence:

$$\chi_1(\mathcal{D})((\mathrm{Id} - P_1)(x)) = \chi_1(\mathcal{D})(\mathrm{Id} - \chi_0(\mathcal{D}))(y) = 0.$$

Therefore $\chi_1(\mathcal{D})P_1 = \chi_1(\mathcal{D})$. Taking the adjoint shows also $P_1\chi_1(\mathcal{D}) = \chi_1(\mathcal{D})$. In particular, we have found a spectral cut χ_1 and a projection P_1 such that $\mathrm{Im}\chi_1(\mathcal{D}) \subset \mathrm{Im}(P_1)$.

We now proceed to modify the projection P_1 to a spectral section of $\mathcal{D}$. Let ψ be a spectral cut such that $\psi(t) = 1$ on $[-1, +\infty[$, we set $\psi_N(t) = \psi(t/N)$. Since $\chi_1(\mathcal{D}) - P_1$ is a $\Lambda-$compact operator, one checks easily that

$$\psi_N(\mathcal{D})(\chi_1(\mathcal{D}) - P_1) \to \chi_1(\mathcal{D}) - P_1.$$

Thus there exists $N_0 > 0$ such that $\psi_{N_0}(t)\chi_1(t) = \chi_1(t)$ and:

$$||P_1 - \chi_1(\mathcal{D}) + \psi_{N_0}(\mathcal{D})(\chi_1(\mathcal{D}) - P_1)|| = ||P_1 - \psi_{N_0}(\mathcal{D})P_1|| < \frac{1}{2}.$$

Applying Lemma 5, we obtain a projection $\mathcal{P} = \mathcal{P}^2 = \mathcal{P}^*$ in $B_\Lambda L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)$ such that

$$\mathrm{Im}(\mathcal{P}) = \psi_{N_0}(\mathcal{D})P_1(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)).$$

Let χ_2 be a smooth spectral cut such that $\chi_2(t)\psi_{N_0}(t) = \psi_{N_0}(t)$. Then $\chi_2(\mathcal{D})\psi_{N_0}(\mathcal{D})P_1 = \psi_{N_0}(\mathcal{D})P_1$ and we have:

$$\mathcal{P}(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f)) = \mathrm{Im}(\chi_2(\mathcal{D})\psi_{N_0}(\mathcal{D})P_1) \subset \mathrm{Im}\chi_2(\mathcal{D}).$$

On the other hand, since $P_1\chi_1(\mathcal{D}) = \chi_1(\mathcal{D})$, we have:

$$\chi_1(\mathcal{D}) = \psi_{N_0}(\mathcal{D})\chi_1(\mathcal{D}) = \psi_{N_0}(\mathcal{D})P_1\chi_1(\mathcal{D}).$$

This implies that:

$$\mathrm{Im}\chi_1(\mathcal{D}) = \mathrm{Im}(\psi_{N_0}(\mathcal{D})P_1\chi_1(\mathcal{D})) \subset \mathrm{Im}(\psi_{N_0}(\mathcal{D})P_1) = \mathrm{Im}\mathcal{P}.$$

Therefore, $\mathrm{Im}\chi_1(\mathcal{D}) \subset \mathrm{Im}\mathcal{P} \subset \mathrm{Im}\chi_2(\mathcal{D})$ and $\mathcal{P}$ is a spectral section of $\mathcal{D}$. Theorem 3 is proved.

Remark. Theorem 3 is the noncommutative extension of Proposition 1 of [27]. The reason why the proof of Theorem 3 (given here) does not follow the lines of a mere noncommutative extension of the proof of Proposition 1 of [27] is that we do not have the following result, which is thus stated as a conjecture. The conjecture is motivated by Theorem B of [1] (which is used in [27]) and by Theorem 1.

Conjecture.
Let $(F_1, \mathcal{E}_1)$ and $(F_2, \mathcal{E}_2)$ be two Kasparov $\Lambda-$modules such that the two $\Lambda-$ Hilbert modules $\mathcal{E}_1, \mathcal{E}_2$ are isomorphic to $l^2(\Lambda)$ and for any $f \in C_c^\infty(]0, +\infty[; [0,1])$ which is equal to one in a neighborhood of 1, $f(F_j)$ and $\mathrm{Id} - f(F_j)$ induce very full projections in the Calkin algebra $(1 \leq j \leq 2)$. Assume that F_1 and F_2 have the same index in $K_1(\Lambda)$. Then $(F_1, \mathcal{E}_1)$ is operator-homotopic equivalent to a $\Lambda-$Kasparov module which is unitary equivalent to $(F_2, \mathcal{E}_2)$.

The content of this result is that under the stated assumptions, it is possible to make the relevant homotopies without adding degenerate modules. If this conjecture is true, then applying it with $F_1 = \frac{\mathcal{D}}{\sqrt{\mathrm{Id} + \mathcal{D}^2}}$ and F_2 an operator whose spectrum admits a gap at zero, it is possible to prove Theorem 3 by following closely the route of [27] pages 107-108.

2.5. **Difference classes and the noncommutative spectral flow.**

We recall the notion of difference class associated to two spectral sections. The definition is already given in [32] [33] and it is an easy and natural extension to the noncommutative setting of the definition given in ([27]). We shall need the following proposition of Wu whose proof (left to the reader) uses the same technique as in the proof of Theorem 3.

Proposition 3. *If there exists a spectral section for $\mathcal{D}$, then for any spectral cut χ_1, there is a spectral cut χ_2 such that $\chi_1\chi_2 = \chi_1$, and a spectral section $\mathcal{R}$ satisfying* $\mathrm{Im}\,\chi_1(\mathcal{D}) \subset \mathrm{Im}\,\mathcal{R} \subset \mathrm{Im}\,\chi_2(\mathcal{D})$. *Similarly, for any spectral cut χ_2, there is a spectral cut χ_1 such that $\chi_1\chi_2 = \chi_1$, and a spectral section $\mathcal{Q}$ satisfying* $\mathrm{Im}\,\chi_1(\mathcal{D}) \subset \mathrm{Im}\,\mathcal{Q} \subset \mathrm{Im}\,\chi_2(\mathcal{D})$.

This Proposition implies immediately the following:

Corollary 1. *Let $\mathcal{P}_1, \mathcal{P}_2$ be two spectral sections of $\mathcal{D}$. Then there exist spectral sections $\mathcal{R}, \mathcal{Q}$ such that for $i = 1, 2$*

$$\mathcal{P}_i\mathcal{R} = \mathcal{R}\mathcal{P}_i = \mathcal{P}_i \text{ and } \mathcal{P}_i\mathcal{Q} = \mathcal{Q}\mathcal{P}_i = \mathcal{Q}.$$

Definition 4. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two spectral sections of $\mathcal{D}$. Then there exists a class $[\mathcal{P}_1 - \mathcal{P}_2] \in K_0(\Lambda)$, called the difference class, which is defined as follows: choose a spectral section $\mathcal{Q}$ of $\mathcal{D}$ satisfying $\mathcal{P}_i\mathcal{Q} = \mathcal{Q} = \mathcal{Q}\mathcal{P}_i$ for $i \in \{1, 2\}$, such a spectral section $\mathcal{Q}$ exists by Corollary 1. Then $\mathcal{P}_1 - \mathcal{Q}$ and $\mathcal{P}_2 - \mathcal{Q}$ are projections in $\mathcal{K}_\Lambda(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f))$ and $[\mathcal{P}_1 - \mathcal{P}_2] := [\mathcal{P}_1 - \mathcal{Q}] - [\mathcal{P}_2 - \mathcal{Q}]$ is well defined as a class in $K_0(\mathcal{K}_\Lambda(L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_f))) = K_0(\Lambda)$.

Remark. The definition is well posed, independent of the choice of Q. The proof of this fact proceeds, *mutatis mutandis*, as in [27], page 127.

Now we recall Wu's definition of noncommutative spectral flow; this is an extension to the noncommutative context of the notion of higher spectral flow given in [10].

Let $(f_u)_{u\in[0,1]}$ be a continuous family of maps $f_u : M \to B\Gamma$. Let $(g_u)_{u\in[0,1]}$, $(h_u)_{u\in[0,1]}$ and $(\nabla^{E,u})_{u\in[0,1]}$ be continuous families of respectively riemannian metrics on M, hermitian metrics on E and Clifford unitary connections on (E, h_u) with respect to g_u for each $u \in [0,1]$. Let $(\mathcal{D}_u)_{u\in[0,1]}$ be the associated continuous family of Dirac type operators, where each $\mathcal{D}_u$ acts on $L^2_\Lambda(M; E\otimes_{\mathbb{C}} \mathcal{V}_{f_u}))$. Recall (see [31], Proposition 6.4.7) that $C^0_o(]0,1]; \mathbb{C})\otimes\Lambda$ is contractible (so that its $K-$Theory groups vanish) and that the following exact sequence (where the last arrow is the evaluation map at 1) splits:

$$0 \to C^0_o(]0,1]; \mathbb{C}) \otimes \Lambda \to C^0([0,1]; \mathbb{C}) \otimes \Lambda \to \Lambda \to 0.$$

Thus, using the six-terms long exact sequence in $K-$Theory, we get an isomorphism

$$\mathcal{U} : K_1(C^0([0,1]; \mathbb{C}) \otimes \Lambda\,) \simeq\, K_1(\Lambda).$$

Now, assume that the index $\delta([\chi(\mathcal{D}_0)])$ vanishes in $K_1(\Lambda)$, then using the above isomorphism $\mathcal{U}$, one gets that $\delta([\chi((\mathcal{D}_u)_{u\in[0,1]})])$ vanishes in $K_1(C^0([0,1])\otimes\Lambda)$. Summarizing: the family $(\mathcal{D}_u)_{u\in[0,1]}$ admits a (total) spectral section $\mathcal{P} = (\mathcal{P}_u)_{u\in[0,1]}$.

Definition 5. If $\mathcal{Q}_0$ (resp. $\mathcal{Q}_1$) is a spectral section associated with $\mathcal{D}_0$ (resp. $\mathcal{D}_1$) then the noncommutative spectral flow $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1)$ from $(\mathcal{D}_0, \mathcal{Q}_0)$ to $(\mathcal{D}_1, \mathcal{Q}_1)$ through $(\mathcal{D}_u)_{u\in[0,1]}$ is the $K_0(\Lambda)-$class:

$$\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1) = [\mathcal{Q}_1 - \mathcal{P}_1] - [\mathcal{Q}_0 - \mathcal{P}_0] \in K_0(\Lambda).$$

This definition does not depend on the particular choice of the total spectral section $\mathcal{P} = (\mathcal{P})_{u\in[0,1]}$. If morever the family is periodic (ie $\mathcal{D}_1 = \mathcal{D}_0$) and if we take $\mathcal{Q}_1 = \mathcal{Q}_0$ then $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_0)$ does not depend on the choice of $\mathcal{Q}_1 = \mathcal{Q}_0$.

Remark. The definition of noncommutative spectral flow, together with some noncommutative extensions of results of Dai-Zhang, have been already published in [18]. In particular, a formula for the Chern character of the noncommutative spectral flow was proved there (Theorem 5.3); the formula involves the higher eta invariant.

Remark. The definition of noncommutative spectral flow can be given for any 1-parameter family of Dirac-type operators, not necessarily associated to a variation of classifying map, of the metrics and of the connection.

2.6. Existence of spectral sections in the even dimensional case.

Now we consider a smooth closed riemannian manifold M of dimension $2m$, a continuous map $f : M \to B\Gamma$ and a complex hermitian $\mathbb{Z}_2-$graded Clifford module $E = E^+ \oplus E^- \to M$ endowed with a unitary Clifford connection. Let $\widetilde{M} \to M$ be the $\Gamma-$normal cover of M associated with $f : M \to B\Gamma$. Let $\mathcal{V}_f = \Lambda \times_\Gamma \widetilde{M} \to M$ be the $\Lambda-$flat bundle associated to these data. Then we denote by $\mathcal{D}$ the $\Lambda-$Dirac type operator acting on $C^\infty(M, E\otimes_{\mathbb{C}} \mathcal{V}_f)$ associated to these data. In fact,

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$$

defines an odd self-adjoint unbounded regular operator acting on the $\Lambda-$Hilbert module $L^2_\Lambda(M, E\otimes_{\mathbb{C}} \mathcal{V}_f)$.

Let τ denotes the grading of $L^2_\Lambda(M, E\otimes_{\mathbb{C}}\mathcal{V}_f)$ induced by the grading-automorphism τ_E defining the splitting $E = E^+ \oplus E^-$ We clearly have $\mathcal{D}\tau = -\tau\mathcal{D}$.

Definition 6. A Cl(1)-spectral section for $\mathcal{D}$ is a spectral section $\mathcal{P}$ with the additional property that

$$\mathcal{P}\tau + \tau\mathcal{P} = \tau\,.$$

Lemma 4.3 of [19] shows the following:

Theorem 4. *$\mathcal{D}$ admits a* Cl(1)*-spectral section $\mathcal{P} \in B_\Lambda(L^2_\Lambda(M, E\otimes_{\mathbb{C}}\mathcal{V}_f))$ if and only if the index of $\mathcal{D}^+$ is zero in $K_0(\Lambda)$.*

3. Manifolds with boundary, spectral sections and index classes

3.1. APS and b-index classes in the even dimensional case.

Let (M, g) be a smooth riemannian manifold of dimension $n = 2m$ with boundary, $E = E^+ \oplus E^-$ a $\mathbb{Z}_2-$graded hermitian Clifford module endowed with a unitary Clifford connection ∇. We assume that on a collar neighborhood ($\simeq [0,1]\times\partial M$) of ∂M, g and ∇ have a product structure. We consider also a finitely generated group Γ, a classifying map $f : M \to B\Gamma$ and denote by Λ the reduced C^*-algebra of Γ. Let $\widetilde{M} \to M$ be the associated $\Gamma-$normal cover. We consider $\mathcal{V}_f = \Lambda \times_\Gamma \widetilde{M}$., a flat bundle over M. These data define a $\mathbb{Z}_2-$graded $\Lambda-$linear Dirac type operator $\mathcal{D}$ acting on $C^\infty(M, \mathcal{V}_f\otimes_{\mathbb{C}} E)$ which, in a collar neightborhood ($\simeq [0,1]\times\partial M$), may be written as:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix},$$

where $\mathcal{D}^+ = \sigma_{\mathcal{D}^+}(dx)(\frac{\partial}{\partial x} + \mathcal{D}_0)$, $\mathcal{D}_0$ is the induced boundary operator acting on the sections of $(\mathcal{V}_f\otimes_{\mathbb{C}} E^+)_{|\partial M}$ and $\mathcal{D}^- = \sigma^-(\frac{\partial}{\partial x} + \sigma^+\mathcal{D}_0\sigma^-)$ where we have set $\sigma^\pm = \sigma_{\mathcal{D}^\pm}(dx)$. We now consider the $\Lambda-$Hilbert modules $L^2_\Lambda(M, \mathcal{V}_f\otimes E)$ and $L^2_\Lambda(\partial M, (\mathcal{V}_f\otimes E^\pm)_{|\partial M})$ defined by the above data. Then the operator $\mathcal{D}_0$ defines a self-adjoint regular unbounded $\Lambda-$linear operator acting on $L^2_\Lambda(\partial M, (\mathcal{V}_f\otimes E)_{|\partial M})$. By the cobordism invariance of the index class (see [18], Proposition 2.3) we know that the $\operatorname{Ind}\mathcal{D}_0^+$ is zero in $K_1(\Lambda)$; thus Theorem 3 shows that $\mathcal{D}_0$ admits spectral section $\mathcal{P} \in BL^2_\Lambda(\partial M, (\mathcal{V}_f\otimes E^+)_{|\partial M})$, moreover ([18], Theorem 2.6 1]), one may assume that $\mathcal{P}$ is a $\Lambda-$pseudodifferential operator of order zero. Then we define an odd operator acting on $L^2_\Lambda(\partial M, (\mathcal{V}_f\otimes E)_{|\partial M})$ by:

$$B_{\mathcal{P}} = \begin{pmatrix} 0 & (\operatorname{Id}-\mathcal{P})\sigma^- \\ \sigma^-\mathcal{P} & 0 \end{pmatrix}$$

Next we introduce the domain $\operatorname{dom}(\mathcal{D}_{\mathcal{P}})$ of $\mathcal{D}$ associated with the global APS-boundary condition defined by $\mathcal{P}$:

$$\operatorname{dom}(\mathcal{D}_{\mathcal{P}}) = \{\xi \in H^1_\Lambda(M, \mathcal{V}_f\otimes E)/B_{\mathcal{P}}(\xi_{|\partial M}) = 0\}$$

and will denote by $\mathcal{D}_{\mathcal{P}}$ the restriction of $\mathcal{D}$ to $\operatorname{dom}(\mathcal{D}_{\mathcal{P}})$. In a similar and obvious way one defines $\mathcal{D}^\pm_{\mathcal{P}}$. We also set:

$$\operatorname{dom}(\mathcal{D}^2_{\mathcal{P}}) = \{\xi \in \operatorname{dom}(\mathcal{D}_{\mathcal{P}})/\mathcal{D}(\xi) \in \operatorname{dom}(\mathcal{D}_{\mathcal{P}})\}.$$

Then Wu has shown in [33], page 174 (this is his published paper) that the heat operator $H(s) = \mathrm{e}^{-s\mathcal{D}^+_{\mathcal{P}}\mathcal{D}^-_{\mathcal{P}}}$ is well defined for s real > 0 as a $\Lambda-$compact operator sending $L^2_\Lambda(M, \mathcal{V}_f\otimes E)$ into $\operatorname{dom}(\mathcal{D}^2_{\mathcal{P}})$ and that $\operatorname{Id}+\mathcal{D}^2_{\mathcal{P}}$ is invertible with an inverse being $\Lambda-$compact. Thus $\mathcal{D}^-_{\mathcal{P}} \circ \int_0^{+\infty} H(s)ds$ is a parametrix for $\mathcal{D}^+_{\mathcal{P}}$ and $\mathcal{D}^+_{\mathcal{P}}$ is indeed a $\Lambda-$Fredholm operator from $\mathrm{dom}(\mathcal{D}^+_{\mathcal{P}})$ to $L^2_\Lambda(M, \mathcal{V}_f \otimes E^-)$; we will denote by $\mathrm{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}) \in K_0(\Lambda)$ the associated higher (APS) index class.

Now we are going to recall the definition of the $b-$higher index class (see [18]) associated to $\mathcal{D}^+$. According to [33], [18] there exists a $\Lambda-$compact self-adjoint operator $\mathcal{A}^0_{\mathcal{P}} \in KL^2_\Lambda(\partial M, (\mathcal{V}_f \otimes E)_{|\partial M})$ such that $\mathcal{D}^+_0 + \mathcal{A}^0_{\mathcal{P}}$ is invertible and

$$\mathcal{P} = \frac{1}{2}(\mathrm{Id} + \frac{\mathcal{D}^+_0 + \mathcal{A}^0_{\mathcal{P}}}{|\mathcal{D}^+_0 + \mathcal{A}^0_{\mathcal{P}}|}).$$

It is proved in [18] (Proposition 2.10) that the operator $\mathcal{A}^0_{\mathcal{P}}$ can be chosen to be a Λ-smoothing operator.

We add to M a cylinder $[0,1] \times \partial M$ and still denote by M the extended manifold. Then we extend the metric g to a $b-$metric having a product structure $(\frac{dx}{x})^2 + g_{\partial M}(y)$ on $[0, 1/2] \times \partial M$ (this amounts to add an infinite cylinder). Then we denote by ${}^b\mathcal{D}$ the $b - \Lambda$-operator associated to $\mathcal{D}$ acting on $L^2_{b,\Lambda}(M, \mathcal{V}_f \otimes E)$ (see Section 11 of [?]). Let $\rho \in C^\infty_c(\mathbb{R}, \mathbb{R}^+)$ be a nonnegative even smooth test function such that $\int_{\mathbb{R}} \rho(x)dx = 1$. We set $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$ and then consider the Fourier transform of ρ_ϵ:

$$\widehat{\rho_\epsilon}(z) = \int_{\mathbb{R}} \mathrm{e}^{-itz} \rho(\frac{t}{\epsilon})\epsilon^{-1} dt.$$

Then there exists an operator $\mathcal{A}_{\mathcal{P}} \in \Psi^{-\infty}_{b,\Lambda}(M, \mathcal{V}_f \otimes E^+, \mathcal{V}_f \otimes E^-)$ such that the indicial family of ${}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}$ is given by:

$$I({}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}, z) = \mathcal{D}_0 + iz + \widehat{\rho_\epsilon}(z)\mathcal{A}^0_{\mathcal{P}}$$

and is invertible for any $z \in \mathbb{R}$ (see MP I and Lemma 6.1 of [18]). So ${}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}$ belongs to the $b-\Lambda-$calculus and is a $\Lambda-$Fredholm operator from $H^1_{b,\Lambda}(M, \mathcal{V}_f \otimes E^+)$ to $L^2_{b,\Lambda}(M, \mathcal{V}_f \otimes E^-)$. We shall denote by $\mathrm{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}) \in K_0(\Lambda)$ the associated higher index class.

Theorem 5. *With the above notations, the following equality holds:*

$$\mathrm{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}) = \mathrm{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}).$$

Proof. Let $\varphi \in C^\infty([0,1];[0,1])$, such that $\varphi(x) = 0$ for $\frac{3}{4} \le x \le 1$ and $\varphi(x) = 1$ for $0 \le x \le \frac{1}{2}$. For each $t \in [0,1]$, one defines two perturbed Dirac type $\Lambda-$linear operators by setting: $\mathcal{D}^+(t) = \sigma^+(\frac{\partial}{\partial x} + \mathcal{D}_0 + t\varphi(x)\mathcal{A}^0_{\mathcal{P}})$ and: $\mathcal{D}^-(t) = \sigma^-(\frac{\partial}{\partial x} + \sigma^+(\mathcal{D}_0 + t\varphi(x)\mathcal{A}^0_{\mathcal{P}})\sigma^-)$. We then set:

$$\mathcal{D}(t) = \begin{pmatrix} 0 & \mathcal{D}^-(t) \\ \mathcal{D}^+(t) & 0 \end{pmatrix}.$$

Wu has constructed (see [33]) for each $s > 0$ a heat operator $\mathrm{e}^{-s\mathcal{D}^2(1)}$ which sends $L^2_\Lambda(M, \mathcal{V}_f \otimes E)$ into $\mathrm{dom}(\mathcal{D}^2_{\mathcal{P}})$. Observe that for each $t \in [0,1]$, $\mathcal{D}^2(1) - \mathcal{D}^2(t)$ is a smoothing operator. Thus, using Duhamel expansion's formula one then defines for each $s > 0$ the heat operator $\mathrm{e}^{-s\mathcal{D}^2(t)}$ which sends $L^2_\Lambda(M, \mathcal{V}_f \otimes E)$ into $\mathrm{dom}(\mathcal{D}^2_{\mathcal{P}})$. Then,

$$\mathcal{D}^-(t) \int_0^{+\infty} \mathrm{e}^{-s\mathcal{D}^+(t)\mathcal{D}^-(t)} ds$$

defines a parametrix for $\mathcal{D}^+(t)$. So the $(\mathcal{D}^+(t))_{t \in [0,1]}$ define a continuously family of $\Lambda-$Fredholm operators from $H^1_\Lambda(M, \mathcal{V}_f \otimes E^+)$ to $L^2_\Lambda(M, \mathcal{V}_f \otimes E^-)$. Thus for any $t \in [0,1]$, $\mathrm{Ind}^{APS}(\mathcal{D}(t)^+, \mathcal{P}) = \mathrm{Ind}^{APS}(\mathcal{D}(1)^+, \mathcal{P})$. Now, following closely the arguments of Section 10 of [15], one easily checks that $\mathcal{D}(1)^+$ induces a $\Lambda-$Fredholm operator, denoted ${}^{\mathrm{cyl}}\mathcal{D}(1)^+$ from $H^1_{b,\Lambda}(M, \mathcal{V}_f \otimes E^+)$ to $L^2_{b,\Lambda}(M, \mathcal{V}_f \otimes E^-)$ and that:

$$\mathrm{Ind}({}^{\mathrm{cyl}}\mathcal{D}(1)^+) = \mathrm{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}) \in K_0(\Lambda)$$
$$\mathrm{Ind}({}^{\mathrm{cyl}}\mathcal{D}(1)^+) = \mathrm{Ind}^{APS}(\mathcal{D}(1)^+, \mathcal{P}).$$

Since we have seen that $\mathrm{Ind}^{APS}(\mathcal{D}(1)^+, \mathcal{P}) = \mathrm{Ind}^{APS}(\mathcal{D}(0)^+, \mathcal{P})$ the Theorem is proved.

By combining the previous Theorem and Proposition 6.1 of [19], i.e. the relative index theorem for b-index classes, one gets at once the following relative index Theorem for APS-index classes:

Theorem 6. *With the above notations, let $\mathcal{P}$, $\mathcal{P}'$ be two spectral sections for $\mathcal{D}_0$. Then one has:*

$$\mathrm{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}) - \mathrm{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}') = [\mathcal{P}' - \mathcal{P}] \in K_0(\Lambda).$$

3.2. APS and b-index classes in the odd dimensional case.

We shall now address the odd-dimensional case. Let (M, g) be a smooth riemannian manifold with boundary of dimension $n = 2m+1$, E a hermitian Clifford module endowed with a unitary Clifford connection ∇. We assume that in a collar neighborhood of ∂M, $(\simeq [0,1] \times \partial M = \{(x,y)\})$, g and ∇ have a product structure. We fix a Clifford action cl_∂ of $T^*\partial M$ on $E_0 = E_{|\partial M}$ by setting for any $\eta \in T^*\partial M$, $cl_\partial(\eta) = cl(dx)cl(\eta)$ where cl denotes the Clifford action of T^*M on E. We set $\tau = \sqrt{-1}cl(dx)$; since $\tau^2 = \mathrm{Id}$, $E_0 = E_0^+ \oplus E_0^-$ is $\mathbb{Z}_2-$graded with $E_0^\pm = \ker(\tau - \pm\,\mathrm{Id})$.

We consider a finitely generated group Γ, and a classifying map $f : M \to B\Gamma$. These data define as before a $\Lambda-$linear Dirac type operator $\mathcal{D}$ acting on $C^\infty(M, \mathcal{V}_f \otimes_{\mathbb{C}} E)$ which, in a collar neighborhood $(\simeq [0,1] \times \partial M)$, may be written as:

$$\mathcal{D} = \sigma_{\mathcal{D}}(dx)(\frac{\partial}{\partial x} + \mathcal{D}_0)$$

where $\mathcal{D}_0$ is the induced boundary $\mathbb{Z}_2-$graded operator acting on the sections of $(\mathcal{V}_f \otimes_{\mathbb{C}} E)_{|\partial M}$. The operator $\mathcal{D}_0$ is a self-adjoint regular unbounded $\Lambda-$linear operator acting on $L^2_\Lambda(\partial M, (\mathcal{V}_f \otimes E)_{|\partial M})$; we know that $\mathrm{Ind}\mathcal{D}_0$ is zero in $K_0(\Lambda)$, so Theorem 4 shows that $\mathcal{D}_0$ admits a $\mathrm{Cl}(1)-$spectral section $\mathcal{P} \in BL^2_\Lambda(\partial M, (\mathcal{V}_f \otimes E)_{|\partial M})$; by definition we have $\tau\mathcal{P} + \mathcal{P}\tau = \tau$. One may assume as in the even case that $\mathcal{P}$ is a $\Lambda-$pseudodifferential operator of order zero. We fix such a $\mathrm{Cl}(1)-$spectral section $\mathcal{P}$ in the rest of this section. We can find (proceeding as in the even-dimensional case) a self-adjoint *odd* $\Lambda-$compact operator $\mathcal{A}^0_{\mathcal{P}}$ acting on $L^2_\Lambda(\partial M, (\mathcal{V}_f \otimes E)_{|\partial M})$ such that $\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}}$ is invertible and

$$\mathcal{P} = \frac{1}{2}(\mathrm{Id} + \frac{\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}}}{|\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}}|}). \tag{18}$$

The fact that $\mathcal{A}^0_{\mathcal{P}}$ be odd is implied by the condition $\tau\mathcal{P} + \mathcal{P}\tau = \tau$, as in [28].

We denote by ${}^b\mathcal{D}$ the $b-$operator induced by $\mathcal{D}$. Proceeding as in [27] (Lemma 9) and in [18], one checks the existence of a self-adjoint operator $\mathcal{A}_{\mathcal{P}} \in \Psi^{-\infty}_{b,\Lambda}(M, \mathcal{V}_f \otimes E, \mathcal{V}_f \otimes E)$ with indicial family given by:

$$I({}^b\mathcal{D} + \mathcal{A}, z) = \mathcal{D}_0 + iz + \widehat{\rho}_\epsilon(z)\mathcal{A}^0_{\mathcal{P}} \tag{19}$$

This indicial family is invertible for any $z \in \mathbb{R}$.

Proposition 4. *The operator* ${}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}}$ *is self-adjoint regular on* $L^2_{b,\Lambda}(M,\mathcal{V}_f\otimes E)$.

Proof. We follow [33] page 174. Proceeding as in [20] (page 328) and as in [27] (Proposition 8) one constructs for any $s>0$ a heat operator $H(s)=\mathrm{e}^{-s({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})^2}$ such that $R=\int_0^{+\infty}\mathrm{e}^{-s}H(s)ds$ sends $L^2_{b,\Lambda}(M,\mathcal{V}_f\otimes E)$ into $H^2_{b,\Lambda}(M,\mathcal{V}_f\otimes E)$ and $(\mathrm{Id}+({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})^2)\circ R=\mathrm{Id}$. Thus $(\mathrm{Id}+({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})^2)$ is surjective. Moreover, for any $\xi,\xi'\in H^1_{b,\Lambda}(M,\mathcal{V}_f\otimes E)$ one has:

$$<({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})(\xi_1);\xi_2>=<\xi_1;({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})(\xi_2)>\ \in\ \Lambda$$

which completes the proof of the Proposition.

Proceeding along the same lines, one proves the following:

Proposition 5. *Let* $\mathcal{D}^{APS}_{\mathcal{P}}$ *be the operator induced by* $\mathcal{D}$ *defined on the domain* $\mathrm{dom}(\mathcal{D}^{APS}_{\mathcal{P}})=\{\xi\in H^1_{b,\Lambda}(M,\mathcal{V}_f\otimes E)/\,\mathcal{P}(\xi_{|\partial M})=0\}$. *Then* $\mathcal{D}^{APS}_{\mathcal{P}}$ *is a self-adjoint unbounded regular operator on* $L^2_{\Lambda}(M,\mathcal{V}_f\otimes E)$.

Definition 7. One defines two suspended families indexed by $t\in[0,\pi]$ by the following two formulae:
1] $(\Sigma({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}}))_t:=\sqrt{\mathrm{Id}+({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})^2}\cos t+\sqrt{-1}({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})\sin t$.
The family $\Sigma({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}})=(\Sigma({}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}}))_{t\in[0,\pi]}$ defines a $(C^0_c(]0,\pi[)\otimes\Lambda)-$linear operator acting on $C^0_c(]0,\pi[)\otimes H^1_{b,\Lambda}(M,\mathcal{V}_f\otimes E)$.
2] $(\Sigma(\mathcal{D}^{APS}_P))_t:=\sqrt{\mathrm{Id}+(\mathcal{D}^{APS}_P)^2}\cos t+\sqrt{-1}\mathcal{D}^{APS}_P\sin t$.

The easy proof of the next Proposition is left to the reader.

Proposition 6.
1] The index class $\mathrm{Ind}(\Sigma({}^b\mathcal{D}+\mathcal{A}))$ *is well defined in* $K_0(C^0_c(]0,\pi[)\otimes\Lambda)\simeq K_1(\Lambda)$.
2] The index class $\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_P)$ *is well defined in* $K_0(C^0_c(]0,\pi[)\otimes\Lambda)\simeq K_1(\Lambda)$.

We can use suspension in order to extend to the odd dimensional case Definition 4. We follow Proposition 4 of [28].

Definition 8. Consider two $\mathrm{Cl}(1)-$spectral sections P_1,P_2 of $\mathcal{D}_0$, let $\mathcal{A}^0_{P_j}$ ($j\in\{1,2\}$) be a $\mathbb{Z}_2-$graded self-adjoint $\Lambda-$compact operator acting on $L^2_{\Lambda}(\partial M;(\mathcal{V}_f\otimes E)_{|\partial M})$ such that for each $j\in\in\{1,2\}$, $\mathcal{D}_0+\mathcal{A}^0_{P_j}$ is invertible and:

$$P_j=\frac{1}{2}(\mathrm{Id}+\frac{\mathcal{D}_0+\mathcal{A}^0_{P_j}}{|\mathcal{D}_0+\mathcal{A}^0_{P_j}|}).$$

Then for any $t\in[0,\pi]$,

$$\mathcal{D}^j_0(t)=\tau(\,\mathrm{Id}+(\mathcal{D}_0+\mathcal{A}^0_{P_j})^2\,)^{1/2}\cos t\ +\ (\mathcal{D}_0+\mathcal{A}^0_{P_j})\sin t$$

is invertible, where $\tau=\sqrt{-1}cl(dx)$ is the involution of $E_{\partial M}$. Let $P_j(t)$ denote the projection onto the positive part of $\mathcal{D}^j_0(t)$. According to Corollary 1), we consider a continuous family $(Q(t))_{t\in[0,\pi]}$ of projections in $B_\Lambda L^2_\Lambda(\partial M;(\mathcal{V}_f\otimes E)_{|\partial M})$ such that for any $t\in[0,\pi]$ and $j\in\{1,2\}$, $P_j(t)Q(t)=Q(t)P_j(t)=Q(t)$. Then

$$[P_1-P_2]=[\,(P_1(t)-P_2(t))_t\,]=[\,(P_1(t)-Q(t))_t\,]-[\,(P_2(t)-Q(t))_t\,]$$

is a well defined class in $K_0(C^0_c(]0,\pi[)\otimes\Lambda)\simeq K_1(\Lambda)$ which does not depend on the choice of $\mathcal{A}^0_{P_j}$ ($j\in\{1,2\}$) and $(Q(t))_{t\in[0,\pi]}$.

Remark. In fact the previous definition of difference class $[P_1 - P_2]$ holds equally well when one replaces ∂M (resp. $f : \partial M \to B\Gamma$) by a closed manifold F (resp. a map $f' : F \to B\Gamma$) which is not necessarily the boundary of a manifold.

There is a third index class that one can consider. Following [28], we are going to associate a *Dirac suspension* to the $\Lambda-$operator ${}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}}$. For any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we denote by L_θ the line bundle over $\mathbb{R}/\pi\mathbb{Z}$ obtained by identifying $(0, v)$ with $(\pi, \exp(-\sqrt{-1}\theta)v)$, $v \in \mathbb{C}$. Notice that the first cicle has length 2π whereas the second circle has length π. We then denote by $\mathcal{L}$ the bundle of $C^0(\mathbb{R}/\pi\mathbb{Z})-$modules over $\mathbb{R}/2\pi\mathbb{Z}$ whose fiber over $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is given by $\mathcal{L}_\theta = C^0(\mathbb{R}/\pi\mathbb{Z}; L_\theta)$ (recall that the $\mathcal{L}_\theta$ are topologically trivial). We set:

$$S\Lambda = C^0(\mathbb{R}/\pi\mathbb{Z}) \otimes \Lambda.$$

Proposition 7. *One can define a Dirac suspension of* ${}^b\mathcal{D}+\mathcal{A}_{\mathcal{P}}$ *which is a* $\mathbb{Z}_2-$*graded* $S\Lambda-$*Fredholm operator,* ${}^b\mathcal{D}^{\Sigma}_{\mathcal{P}}$*:*

$$H^1_{b,S\Lambda}(\mathbb{R}/2\pi\mathbb{Z} \times M; \mathcal{L} \otimes_{\mathbb{C}} \mathcal{V}_f \otimes E \otimes \mathbb{C}^2) \to L^2_{b,S\Lambda}(\mathbb{R}/2\pi\mathbb{Z} \times M; \mathcal{L} \otimes_{\mathbb{C}} \mathcal{V}_f \otimes E \otimes \mathbb{C}^2)$$

which thus has a well defined index class $\mathrm{Ind}({}^b\mathcal{D}^{\Sigma}_{\mathcal{P}}) \in K_0(S\Lambda)$.

Proof. We follow Section 5 of [28] and explain the modifications that are needed in the present noncommutative context. We introduce the following operator acting on $C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathcal{L})$:

$$\partial^L = -\sqrt{-1}\frac{\partial}{\partial\theta} + \sqrt{-1}\frac{t - \frac{1}{2}}{\pi}.$$

Then, for each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $n \in \mathbb{Z}$, we consider the section $e_n(t,\theta)$ of L_θ defined by $e_n(t,\theta) = \mathrm{e}^{\sqrt{-1}\,n\theta}$ if $t \in [0, \pi[$ and $e_n(t,\theta) = \mathrm{e}^{\sqrt{-1}(n-1)\theta}$ if $t = \pi$. Next, we fix $\epsilon > 0$ and $\delta > 0$ two small (enough) reals and a function $f_{\epsilon,\delta} \in C^\infty_c([0,\pi] \times \mathbb{R})$ which is equal to zero on $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ and equal to one on $[\delta, \pi - \delta] \times [\frac{-1}{\epsilon}, \frac{1}{\epsilon}]$. Then we introduce the $C^0(\mathbb{R}/\pi\mathbb{Z})-$linear operator $R_{\epsilon,\delta}$ acting on $C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathcal{L})$ and defined by the following Schwartz kernel:

$$K(R_{\epsilon,\delta})(t,\theta,t',\theta') = \sum_{n\in\mathbb{Z}} f_{\epsilon,\delta}(t, n + \frac{(t-1/2)}{\pi})\delta_{t=t'} \otimes (e_n(t,\theta) \otimes \overline{e_n(t,\theta')}).$$

Then the required Dirac suspension ${}^b\mathcal{D}^{\Sigma}_{P}$ is defined to be:

$$\begin{pmatrix} 0 & \partial^L \otimes \mathrm{Id} + \mathrm{Id} \otimes \sqrt{-1}\mathcal{D} + R_{\epsilon,\delta} \otimes \sqrt{-1}\mathcal{A} \\ \partial^L \otimes \mathrm{Id} - \mathrm{Id} \otimes \sqrt{-1}\mathcal{D} - R_{\epsilon,\delta} \otimes \sqrt{-1}\mathcal{A} & 0 \end{pmatrix}.$$

This operator is the sum of a Dirac b-$S\Lambda$-operator and a b-$S\Lambda$ operator of order $-\infty$; moreover its indicial family is invertible for $z \in \mathbb{R}$; it follows from [18] that the operator is $S\Lambda$-Fredholm. The Proposition is proved.

In the next Theorem we relate the three index classes introduced above; we also state the relative index theorem in the odd-dimensional case. We devote the next subsection to its proof.

Theorem 7. *With the notation introduced above we have:*
1] $\mathrm{Ind}(\Sigma({}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}})) = \mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}}))$
2] Let $j : K_0(C^0_c(]0,\pi[) \otimes \Lambda) \to K_0(S\Lambda)$ *be the map induced by the inclusion* $]0,\pi[\to \mathbb{R}/\pi\mathbb{Z}$*. Then*

$$j(\mathrm{Ind}(\Sigma({}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}}))) = \mathrm{Ind}({}^b\mathcal{D}^{\Sigma}_{\mathcal{P}}) \in K_0(S\Lambda)\,.$$

3] Let $\mathcal{P}_1, \mathcal{P}_2$ be two $\mathrm{Cl}(1)-$*spectral sections for $\mathcal{D}_0$, then:*

$$\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_1}) - \mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_2}) = [\mathcal{P}_2 - \mathcal{P}_1] \in K_1(\Lambda).$$

We can also give the odd dimensional version of definition 5 thus introducing the noncommutative spectral flow for an odd dimensional manifold F endowed with a $\mathbb{Z}_2-$graded hermitian Clifford module E.

Let $(f_u)_{u\in[0,1]}$ be a continuous family of maps $f_u : F \to B\Gamma$. Let $(g_u)_{u\in[0,1]}$, $(h_u)_{u\in[0,1]}$, and $(\nabla^{E,u})_{u\in[0,1]}$ be continuous families of respectively riemannian metrics on F, hermitian metrics on E, and Clifford unitary connections on (E, h_u) with respect to g_u for each $u \in [0,1]$. Let $(\mathcal{D}_u)_{u\in[0,1]}$ be the associated continuous family of odd Dirac type operators (as above) where each $\mathcal{D}_u$ acts on $L^2_\Lambda(M; E \otimes_{\mathbb{C}} \mathcal{V}_{f_u}))$. Assume that the index $\mathrm{Ind}\,\mathcal{D}_0^+$ vanishes in $K_0(\Lambda)$. Then $\mathrm{Ind}(\mathcal{D}_u^+)_{u\in[0,1]})$ vanishes in $K_0(C^0([0,1])\otimes\Lambda)$ and thus $(\mathcal{D}_u)_{u\in[0,1]}$ admits a $\mathrm{Cl}(1)-$spectral section $(\mathcal{P}_u)_{u\in[0,1]}$.

Definition 9. If $\mathcal{Q}_0$ (resp. $\mathcal{Q}_1$) is a $\mathrm{Cl}(1)-$spectral section associated with $\mathcal{D}_0$ (resp. $\mathcal{D}_1$) then the noncommutative spectral flow $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1)$ from $(\mathcal{D}_0, \mathcal{Q}_0)$ to $(\mathcal{D}_1, \mathcal{Q}_1)$ through $(\mathcal{D}_u)_{u\in[0,1]}$ is the $K_1(\Lambda)-$class:

$$\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1) = [\mathcal{Q}_1 - \mathcal{P}_1] - [\mathcal{Q}_0 - \mathcal{P}_0] \in K_1(\Lambda).$$

This definition does not depend on the particular choice of the spectral section $(\mathcal{P}_u)_{u\in[0,1]}$. If moreover the family is periodic (ie $\mathcal{D}_0 = \mathcal{D}_1$) and if we take $\mathcal{Q}_0 = \mathcal{Q}_1$ then $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_0)$ does not depend on the choice of $\mathcal{Q}_0 = \mathcal{Q}_1$.

Remark. As in the even case, the definition of noncommutative spectral flow can be given for any 1-parameter family of Dirac-type operators.

3.3. **Proof of Theorem 7.**

This Section can be skipped at first reading.

1] Our goal is to show that $\mathrm{Ind}(\Sigma({}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}})) = \mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}}))$. We adapt to the noncommutative setting the proof of Proposition 5 of [28]; we shall also employ arguments used in a similar Proposition proved in [15] (Theorem 10.1) Let $\varphi \in C^\infty([0,1];[0,1])$ such that $\varphi(x) = 1$ for $0 \le x \le 1/2$ and $\varphi(x) = 0$ for $3/4 \le x$. By considering the family of operators:

$$\mathcal{D}(t) = \sigma_{\mathcal{D}}(\frac{\partial}{\partial x} + \mathcal{D}_0 + t\varphi(x)\mathcal{A}^0_{\mathcal{P}})$$

and using an homotopy argument one can replace, in the definition of $\mathcal{D}^{APS}_{\mathcal{P}}$, $\mathcal{D}$ by $\mathcal{D}(1)$. We shall denote by $\mathcal{D}^{APS}_{\mathcal{P}}(1))$ the associated self-adjoint $\Lambda-$Fredholm operator. Then let $\mu > 0$ such that $[-\mu,\mu] \cap \mathrm{spec}(\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}}) = \emptyset$. We observe that we do not change the index class if, in Definition 7, we take instead of $((\Sigma(\mathcal{D}^{APS}_{\mathcal{P}}))_t)$ the family: $(\mathcal{D}^{APS}_{\mathcal{P}}(1) + s\sqrt{-1}\,\mathrm{Id})_{s\in[-\mu/2,\mu/2]}$. Now, for any $s \in [-\mu/2, \mu/2]$ we set:

$$\mathcal{P}_{s\sqrt{-1}} = \frac{1}{2}(\mathrm{Id} + \frac{\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}} + s\sqrt{-1}\tau}{|\mathcal{D}_0 + \mathcal{A}^0_{\mathcal{P}} + s\sqrt{-1}\tau|}).$$

Proceeding as in [28] page 303, using the family $(\mathcal{P}_{ts\sqrt{-1}})_{t\in[0,1]}$ and an homotopy argument, one checks easily the following:

Lemma 6. $\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}}))$ *coincides with the index class defined by the suspended family (parametrized by $s \in [-\mu/2, \mu/2]$):*

$$\mathcal{D}^{APS}_{\mathcal{P}}(1) + s\sqrt{-1}\,\mathrm{Id} : \mathrm{Dom}_s \to L^2_\Lambda(M, \mathcal{V}_f \otimes E)$$

$$\mathrm{Dom}_s = \{u \in H^1_\Lambda(M, \mathcal{V}_f \otimes E) / \mathcal{P}_{s\sqrt{-1}}(\xi_{|\partial M}) = 0\}.$$

Now we extend M to a manifold M_e, with cylindrical end and the boundary defining functions related by $x = \exp(\widetilde{x})$. Let us fix $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(x) = 1$ for $x < 1$ and $\chi(x) \in C^\infty(M_e)$ has support in the collar neighborhood of the boundary. Following closely [28] page 303, we embedd Dom_s into $H^1_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E)$ and obtain the decomposition:

$$\mathrm{Dom}_s \oplus G_s = H^1_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E),$$

where

$$G_s = \{u \in H^1_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E) /\ \mathcal{P}_{s\sqrt{-1}}(u_{|x=1}) = u_{|x=1} \text{ and } \forall x > 1\ u = \chi(x) u_{|x=1}\}.$$

Moreover, $\mathcal{D}(1)$ induces an operator, denoted ${}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}}$ from $H^1_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E)$ to $L^2_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E)$. Then, proceeding as in [28] page 303-304, one checks easily that for $s \in]-\mu, \mu[$, ${}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}} - \sqrt{-1}s\,\mathrm{Id}$ induces an invertible operator from G_s onto its range denoted H_s and that

$$L^2_{b,\Lambda}(M_e, \mathcal{V}_f \otimes E) = L^2(M, \mathcal{V}_f \otimes E) \oplus H_s.$$

From this and Lemma 6, we conclude that $\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}})) = \mathrm{Ind}(\Sigma({}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}}))$. Lastly, an homotopy argument (see Section 10 of [15]) shows that: $\mathrm{Ind}(\Sigma({}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}})) = \mathrm{Ind}(\Sigma({}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}}))$ which completes the proof of 1].

2] Let $j : K_0(C^0_c(]0,\pi[) \otimes \Lambda) \to K_0(S\Lambda)$ be the map induced by the inclusion $]0,\pi[\to \mathbb{R}/\pi\mathbb{Z}$. Our goal is to prove that $j(\mathrm{Ind}(\Sigma({}^b\mathcal{D} + \mathcal{A}_{\mathcal{P}}))) = \mathrm{Ind}({}^b\mathcal{D}^{\Sigma}_{\mathcal{P}}) \in K_0(S\Lambda)$.. We observe that $L^2_{b,S\Lambda}(\mathbb{R}/2\pi\mathbb{Z} \times M; \mathcal{L} \otimes_{\mathbb{C}} \mathcal{V}_f \otimes E \otimes \mathbb{C}^2)$ is isomorphic to the field of continuous $\Lambda-$Hilbert modules over $\mathbb{R}/\pi\mathbb{Z}$ given by: $\oplus_{n\in\mathbb{Z}} \mathbb{C}e_n(t,.) \otimes L^2_{b,\Lambda}(M; \mathcal{V}_f \otimes E \otimes \mathbb{C}^2)$. Now an easy translation in the noncommutative setting of the proof of the Proposition 6 of [28] (pages 308-309) allows to finish the proof of 2]

3] We shall now prove the relative index theorem in the odd dimensional case: $\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_1}) - \mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_2}) = [\mathcal{P}_2 - \mathcal{P}_1] \in K_1(\Lambda)$. We follow the proof of Proposition 11 of [28] and explain the modifications allowing to extend it to our noncommutative setting. We fix three real numbers, $a < 0$, $b > 1$ and $\epsilon > 0$. We consider a function $\psi_\epsilon(t,\lambda) \in C^\infty([0,\pi) \times \mathbb{R})$ which vanishes on the complement of the open rectangle $]0,\pi[\times]a, a + b\pi[$ and is equal to one on the rectangle $[\epsilon, \pi - \epsilon] \times [a + \epsilon, a + b\pi - \epsilon]$. Next we introduce the $C^0(\mathbb{R}/\pi\mathbb{Z})-$linear operator R_ϵ acting on $C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathcal{L})$ and defined by the following Schwartz kernel:

$$K(R_\epsilon(t,\theta,t',\theta') = \sum_{n\in\mathbb{Z}} \psi_\epsilon(t, n + \frac{(t-1/2)}{\pi})\delta_{t=t'} \otimes (e_n(t,\theta) \otimes \overline{e_n(t,\theta')}).$$

We fix two operators $\mathcal{A}^0_{\mathcal{P}_j}$ $(j \in \{1,2\})$ as in Definition 8 and we associate to them two operators $\mathcal{A}^0_{\mathcal{P}_j}$ as in (19). Then, we consider the Dirac suspension ${}^b\mathcal{D}^{\Sigma}_{\mathcal{P}_j}$ defined by

$$\begin{pmatrix} 0 & \partial^L \otimes \mathrm{Id} + \mathrm{Id} \otimes \sqrt{-1}\mathcal{D} + R_\epsilon \otimes \sqrt{-1}\mathcal{A}_{\mathcal{P}_j} \\ \partial^L \otimes \mathrm{Id} - \mathrm{Id} \otimes \sqrt{-1}\mathcal{D} - R_\epsilon \otimes \sqrt{-1}\mathcal{A}_{\mathcal{P}_j} & 0 \end{pmatrix}.$$

Next we observe that $L^2_{b,S\Lambda}(\mathbb{R}/2\pi\mathbb{Z} \times \partial M; \mathcal{L} \otimes_{\mathbb{C}} \mathcal{V}_f \otimes E_{|\partial M})$ is isomorphic to the field of continuous $\Lambda-$Hilbert modules over $\mathbb{R}/\pi\mathbb{Z}$ given by $V_0(t) \oplus V(t)$ where:

$$V_0(t) = \mathbb{C}e_0(t,.) \otimes L^2_\Lambda(\partial M; \mathcal{V}_f \otimes E_{|\partial M})$$

$$V(t) = \oplus_{n\in\mathbb{Z}\setminus\{0\}}\mathbb{C}e_n(t,.)\otimes L^2_\Lambda(\partial M;\mathcal{V}_f\otimes E_{|\partial M})$$

where we have written $\mathcal{V}_f$ instead of $(\mathcal{V}_f)_{|\partial M}$.

Now we consider the two following families of operators:

$$(\mathrm{Id}+(\mathcal{D}_0+\mathcal{A}^0_{\mathcal{P}_j})^2)^{1/2}\tau\cos t+(\mathcal{D}_0+\mathcal{A}^0_{\mathcal{P}_j})\sin t \tag{20}$$

(21)

$$(\mathrm{Id}+(\mathcal{D}_0+\psi_\epsilon(t,\frac{(t-1/2)}{\pi})\mathcal{A}^0_{\mathcal{P}_j})^2)^{1/2}\tau\cos t+(\mathcal{D}_0+\psi_\epsilon(t,\frac{(t-1/2)}{\pi})\mathcal{A}^0_{\mathcal{P}_j})\sin t,$$

where $j\in\{1,2\}$. Observe that for each $t\in(0,\pi]$, the operators of (20) and (21) are invertible, then by considering the projections onto their positive part, one defines the class $[\mathcal{P}^\epsilon_2-\mathcal{P}^\epsilon_1]\in K_1(\Lambda)$ exactly as in Definition 8. Proceeding as in [28] (pages 316-317), one checks that:

$$[\mathcal{P}^\epsilon_2-\mathcal{P}^\epsilon_1]=\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_1})-\mathrm{Ind}(\Sigma(\mathcal{D}^{APS}_{\mathcal{P}_2}).$$

Since the (all invertible) operators in (20) and (21) are invertible and coincide for at least one real $t\in]1/2,\pi]$, we have $[\mathcal{P}^\epsilon_2-\mathcal{P}^\epsilon_1]=[\mathcal{P}_2-\mathcal{P}_1]$.
The Theorem is now completely proved.

4. A splitting formula for index classes

4.1. The even dimensional case.

Our first task consists in computing higher APS-index classes for the cylinder in the even-dimensional case.

Let N be a closed manifold of dimension $2m+1$, $f:N\to B\Gamma$ a continuous map and $E\to N$ be a Clifford hermitian module over N endowed with a unitary Clifford connection. Let $\mathcal{D}_N$ be the associated $\Lambda-$Dirac type operator acting on $L^2_\Lambda(N,\mathcal{V}_f\otimes E)$. We fix a constant map $c:[-1,1]\to B\Gamma$. Let $\mathcal{D}$ be the associated $\mathbb{Z}_2-$graded Dirac type operator acting on $L^2_\Lambda([-1,1]\times N,\mathcal{V}_{f\times c}\otimes E\otimes\mathbb{C}^2)$, with boundary operator equal to $\mathcal{D}_N$. We fix a spectral section $\mathcal{P}$ at $\{1\}\times N$ for the boundary operator $\mathcal{D}_N$ (of $\mathcal{D}$) and consider the spectral section $\mathrm{Id}-\mathcal{P}$ at $\{-1\}\times N$.

Lemma 7.

$$\mathrm{Ind}^{APS}(\mathcal{D},\mathrm{Id}-\mathcal{P},\mathcal{P})=0\in K_0(\Lambda).$$

Proof. Let $\mathcal{A}^0_{\mathcal{P}}$ be a self-adjoint $\Lambda-$compact operator such that $\mathcal{D}_N+\mathcal{A}^0_{\mathcal{P}}$ is invertible and $\mathcal{P}$ is the projection onto to the positive part of $\mathcal{D}_N+\mathcal{A}^0_{\mathcal{P}}$. We consider the operator ${}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}}$ defined by:

$$\begin{pmatrix} 0 & {}^{\mathrm{cyl}}\mathcal{D}^-_{\mathcal{P}} \\ {}^{\mathrm{cyl}}\mathcal{D}^+_{\mathcal{P}} & 0\end{pmatrix}$$

where ${}^{\mathrm{cyl}}\mathcal{D}^+_{\mathcal{P}}=cl(\frac{dx}{x-1})(\frac{\partial}{\partial x}+\mathcal{D}_N+\mathcal{A}^0_{\mathcal{P}})$ and ${}^{\mathrm{cyl}}\mathcal{D}^-_{\mathcal{P}}$ is the adjoint of ${}^{\mathrm{cyl}}\mathcal{D}^+_{\mathcal{P}}$. Then, proceeding as in Section 10 of [15], one checks that ${}^{\mathrm{cyl}}\mathcal{D}_{\mathcal{P}}$ is $\Lambda-$Fredholm from $H^1_{b,\Lambda}([-1,1]\times N,\mathcal{V}_{f\times c}\otimes E\otimes\mathbb{C}^2)$ to $L^2_{b,\Lambda}([-1,1]\times N,\mathcal{V}_{f\times c}\otimes E\otimes\mathbb{C}^2)$ and that $\mathrm{Ind}^{APS}(\mathcal{D},\mathrm{Id}-\mathcal{P},\mathcal{P})=\mathrm{Ind}({}^{\mathrm{cyl}}\mathcal{D}^+_{\mathcal{P}})$. But using Mellin transform on the cylinder one sees that ${}^{\mathrm{cyl}}\mathcal{D}^2_{\mathcal{P}}$ is invertible, because $\xi^2+(\mathcal{D}_N+\mathcal{A}^0_{\mathcal{P}})^2$ is invertible on $L^2(\mathbb{R},d\xi)\otimes L^2_\Lambda(N,\mathcal{V}_f\otimes E)$. So the index class is zero, which proves the Lemma.

Lemma 8. *We use the notations of the previous Lemma. Let P and Q be two spectral sections for $\mathcal{D}_N$ at $\{1\} \times N$. Then:*

$$\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - Q, P) = [Q - P] \in K_0(\Lambda).$$

Proof. Theorem 7 implies that:

$$\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{Q}) - \mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{P}) = [\mathcal{Q} - \mathcal{P}].$$

Since Lemma 7 implies that $\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{Q}) = 0$ one gets immediately the Lemma.

Now we are going to prove the additivity result for index classes; before stating it, we introduce suitable notations. Let (M, g) be a smooth closed riemannian manifold dimension $n = 2m$, $E = E^+ \oplus E^-$ a $\mathbb{Z}_2-$graded hermitian Clifford module endowed with a unitary Clifford connection ∇. We consider also a finitely generated group Γ, a classifying map $f : M \to B\Gamma$ and denote by Λ the reduced C^*-algebra of Γ. Let $\widetilde{M} \to M$ be the associated $\Gamma-$normal cover. We consider the following flat bundle over M:

$$\mathcal{V}_f = \Lambda \times_\Gamma \widetilde{M}.$$

These data define a $\mathbb{Z}_2-$graded $\Lambda-$linear Dirac type operator $\mathcal{D}$ acting on $L^2_\Lambda(M, \mathcal{V}_f \otimes E)$. This operator has a well defined index class $\mathrm{Ind}(\mathcal{D}^+) \in K_0(\Lambda)$. Let F be a closed cutting hypersurface of M such that $M = M_+ \cup M_-$ where $M_\pm$ are two manifolds whose (common) boundary is F. We assume that all these data have a product structure near F. Let $\mathcal{P}$ and $\mathcal{Q}$ be two spectral sections for the boundary operator of the operator induced by the restriction $\mathcal{D}_{|M_\pm}$ of $\mathcal{D}$ to $M_\pm$. Then we have:

Theorem 8. $\mathrm{Ind}\,(\mathcal{D}^+) = \mathrm{Ind}^{APS}\,(\mathcal{D}^+_{|M_+}, \mathcal{P}) + \mathrm{Ind}^{APS}(\mathcal{D}^+_{|M_-}, \mathrm{Id} - \mathcal{Q}) + [\mathcal{P} - \mathcal{Q}].$

Proof. One just has to adapt to the noncommutative setting (and the even dimensional case) the proof of Theorem 3.1 of [9]. Thus following an idea of Bunke [6], one considers a unitary map $W = a + b + c - d$ from

$$L^2_\Lambda(M, \mathcal{V}_f \otimes E) \oplus L^2_\Lambda([-1, 1] \times F, (\mathcal{V}_f \otimes E)_{[-1,1]\times F})$$

to

$$L^2_\Lambda(M_+ \cup [0, 1] \times F, (\mathcal{V}_f \otimes E)_{M_+\cup[0,1]\times F}) \oplus L^2_\Lambda(M_- \cup [-1, 0] \times F, (\mathcal{V}_f \otimes E)_{M_-\cup[-1,0]\times F})$$

where a, b, c, and d are defined as in [9] page 315. Then one finishes the proof as in [9] by using Lemma 8 instead of Theorem 3.3 of [9]. We omit the (easy) details.

Remark. The formula $\mathrm{Ind}\,(\mathcal{D}^+) = \mathrm{Ind}^{APS}\,(\mathcal{D}^+_{|M_+}, \mathcal{P}) + \mathrm{Ind}^{APS}(\mathcal{D}^+_{|M_-}, \mathrm{Id} - \mathcal{P})$, is a *direct* consequence of the b-pseudodifferential calculus on Galois coverings [17] and of the ideas of Bunke (and Dai-Zhang). That much was known to us since the appearence of [9] and of [32], [33]; the lack of *geometric* applications of such a formula has held us from publishing the result until now; the applications we shall give, in Section 5, to the cut-and-paste invariance of higher signatures seem to be a good enough motivation to finally write down the proof.

4.2. **The odd dimensional case.**

Our next task consists in computing higher APS-index classes for the cylinder in the odd dimensional case. Let N be a closed manifold of dimension $2m$, $f : N \to B\Gamma$ a continuous map and $E \to N$ be a $\mathbb{Z}_2-$graded Clifford hermitian module over N endowed with a unitary Clifford connection. Let $\mathcal{D}_N$ be the associated $\Lambda-$Dirac type operator acting on $L^2_\Lambda(N, \mathcal{V}_f \otimes E$. We fix a constant map $c : [-1,1] \to B\Gamma$. Let $\mathcal{D}$ be the associated $\mathbb{Z}_2-$graded Dirac type operator acting on $L^2_\Lambda([-1,1]\times N, \mathcal{V}_{f\times c}\otimes E \otimes \mathbb{C}^2)$, with boundary operator equal to $\mathcal{D}_N$. We fix a $\mathrm{Cl}(1)-$spectral section $\mathcal{P}$ at $\{1\}\times N$ for the boundary operator $\mathcal{D}_N$ (of $\mathcal{D}$) and consider the $\mathrm{Cl}(1)-$spectral section $\mathrm{Id} - \mathcal{P}$ at $\{-1\}\times N$.

The proof of the following Lemma is basically the same as the one of Lemma 7 and will be omitted.

Notation. In order to simplify the notation we denote by $\mathrm{Ind}^{APS}(\mathcal{D},\mathcal{P})$ the K_1-index class appearing in the statement of Theorem 7.

Lemma 9.

$$\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{P}, \mathcal{P}) = 0 \in K_0(C^0_c(]0,\pi[) \otimes \Lambda) \simeq K_1(\Lambda).$$

Lemma 10. *We use the notations of the previous Lemma. Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathrm{Cl}(1)-$spectral sections for $\mathcal{D}_N$ at $\{1\}\times N$. Then:*

$$\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{P}) = [\mathcal{Q} - \mathcal{P}] \in K_1(\Lambda).$$

Proof. Theorem 7 3] implies that:

$$\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{Q}) - \mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{P}) = [\mathcal{Q} - \mathcal{P}].$$

Since Lemma 9 implies that $\mathrm{Ind}^{APS}(\mathcal{D}, \mathrm{Id} - \mathcal{Q}, \mathcal{Q}) = 0$ one gets immediately the Lemma.

Let (M, g) be a smooth closed riemannian manifold dimension $n = 2m+1$, E a hermitian Clifford module endowed with a unitary Clifford connection ∇. As in the even case we consider a finitely generated group Γ, a classifying map $f : M \to B\Gamma$ and denote by Λ the reduced C^*-algebra of Γ and by

$$\mathcal{V}_f = \Lambda \times_\Gamma \widetilde{M}.$$

the associated flat bundle. These data define a $\Lambda-$linear unbounded self-adjoint Dirac type operator $\mathcal{D}$ acting on $L^2_\Lambda(M, (\mathcal{V}_f \otimes E))$. This operator has a well defined index class $\mathrm{Ind}(\mathcal{D}) \in K_1(\Lambda)$; this class is defined by suspension. Let F be a closed cutting hypersurface of M such that $M = M_+ \cup M_-$ where $M_\pm$ are two manifolds whose F has (common) boundary is F. We assume that all these data have a product structure near F. Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathrm{Cl}(1)-$spectral sections for the boundary operator of the operator induced by the restriction $\mathcal{D}_{|M_\pm}$ of $\mathcal{D}$ to $M_\pm$. Then we have:

Theorem 9. $\mathrm{Ind}\,(\mathcal{D}) = \mathrm{Ind}^{APS}\,(\mathcal{D}_{|M_+}, \mathcal{P}) + \mathrm{Ind}^{APS}(\mathcal{D}_{|M_-}, \mathrm{Id} - \mathcal{Q}) + [\mathcal{P} - \mathcal{Q}].$

Proof. Once again the proof of Theorem 3.1 of [9] (which follows an idea of Bunke [6]) extends without problems; we simply have to use Lemma 10 instead of Theorem 3.3 of [9]. We omit the details.

5. Cut and paste invariance for higher signatures

5.1. A defect formula.

Let (M_i, r_i) $(1 \le i \le 2)$ be a closed oriented smooth manifold endowed with a continuous map $r_i : M_i \to B\Gamma$.

Definition 10. [14] We say that (M_1, r_1) and (M_2, r_2) are SK-equivalent if there exist decompositions

$$M_1 = M_+ \cup_{(F,\phi_1)} M_- \qquad M_2 = M_+ \cup_{(F,\phi_2)} M_-$$

with

$$\partial M_+ = F = -\partial M_- \,, \qquad \phi_i \in \mathrm{Diff}(F)$$

and such that $(r_1)_{|M_\pm} \simeq (r_2)_{|M_\pm}$ ($\simeq$ meaning homotopy equivalence). F is, by assumption, an embedded hypersurface in M_1 and in M_2 which is endowed with the orientation induced by the one of M_+ and the inward normal vector to ∂M_+. For simplicity we assume that F is two-sided (i.e. the normal bundle to F is trivial).

Let (M_i, r_i) $(1 \le i \le 2)$ be as in Definition 10. We denote by $\widetilde{M_i}$ the coverings defined by r_i. Thus $\widetilde{M_i} := (r_i)^* E\Gamma$. We also let $\widetilde{F_i} \to F$ be the covering associated with $(r_i)_{|F} : F \to B\Gamma$. We consider the flat bundles $\mathcal{V}_i = \widetilde{M_i} \times_\Gamma \Lambda$ over M. We now introduce riemannian metrics g_i on M_i and endow the coverings $\widetilde{M_i}$ with the lifted metrics. We denote by $\mathcal{D}^{\mathrm{sign},r_i}_{M_i}$ the associated signature operator on M_i with coefficients in $\mathcal{V}_i$. The restriction of this operator to $M_\pm$ will be denoted by $\mathcal{D}^{\mathrm{sign},r_i}_{M_\pm}$; we use the precise notation $\mathcal{D}^{\mathrm{sign},r_i}_{\partial M_\pm}$ for the associated boundary operators.

Our goal in this Section is to express the difference

$$\mathrm{Ind}\,\mathcal{D}^{\mathrm{sign},r_2}_{M_2} - \mathrm{Ind}\,\mathcal{D}^{\mathrm{sign},r_1}_{M_1} \in K_*(C^*_r(\Gamma))$$

in terms of a noncommutative spectral flow, on F, naturally associated to the above geometric data.

First we need a general result (see [10] for the family case). Let X be an even-dimensional riemannian manifold with boundary and let E be a $\mathbb{Z}_2$-graded complex vector bundle over X. We assume that there exists a 1-parameter family of riemannian metrics $\{g_u\}_{u\in[1,2]}$ on X and a 1-parameter family of hermitian metrics $\{h^E_u\}_{u\in[1,2]}$ on E so that (E, h_u) is a unitary $\mathbb{Z}_2$-graded Clifford module for (X, g_u). We also assume that there is a 1-parameter family of connections $\nabla^{E,u}$ that are unitary with respect to h^E_u and Clifford with respect to the Levi-Civita connection associated to g_u. We assume that these data depend continuously on $u \in [1,2]$. Let $\not{D}_u$ be the Dirac operator associated to $(g_u, h^E_u, \nabla^{E,u})$. Let $f_u : X \to B\Gamma$, $u \in [1,2]$ be a 1-parameter family of continuous maps. Let $\{\not{\mathcal{D}}_u\}_{u\in[1,2]}$, $\not{\mathcal{D}}_u : \mathcal{C}^\infty(X, E\otimes\mathcal{V}_u) \to \mathcal{C}^\infty(X, E\otimes\mathcal{V}_u)$, be the resulting family of $C^*_r(\Gamma)$-linear operators.

Proposition 8. *Let us denote by $\{(\not{\mathcal{D}}_u)_0\}$ the family of boundary-operators associated to $\{\not{\mathcal{D}}_u,\, u \in [1,2]\}$. We fix noncommutative spectral sections $\mathcal{P}_1$, $\mathcal{P}_2$ for $(\not{\mathcal{D}}_1)_0$ and $(\not{\mathcal{D}}_2)_0$ respectively. Then*

$$\mathrm{Ind}^{APS}(\not{\mathcal{D}}^+_2, \mathcal{P}_2) - \mathrm{Ind}^{APS}(\not{\mathcal{D}}^+_1, \mathcal{P}_1) = \mathrm{sf}(\{(\not{\mathcal{D}}_u)_0\}; \mathcal{P}_2, \mathcal{P}_1) \quad \textit{in} \quad K_0(C^*_r(\Gamma)) \tag{22}$$

Proof. We are going to extend to the noncommutative setting the proof of Theorem 5.2 of [10]. Let $(Q(u))_{u\in[1,2]}$ be a total spectral section for the family $\{(\not{\mathcal{D}}_u)_0,\, u \in [1,2]\}$. Then we get a continuous family of elliptic boundary value problems $(\not{\mathcal{D}}_u, Q(u))$, $1 \le u \le 2$. By the homotopy invariance of the K-theoretic index class we get:

$$\mathrm{Ind}(\not{\mathcal{D}}^+_1, Q(1)) = \mathrm{Ind}((\not{\mathcal{D}}^+_2, Q(2)).$$

By the relative index Theorem 6 we get:

$$\mathrm{Ind}(\not{\mathcal{D}}_1^+, Q(1)) = \mathrm{Ind}(\not{\mathcal{D}}_1^+, P_1) + [\mathcal{P}_1 - Q(1)]$$

and

$$\mathrm{Ind}(\not{\mathcal{D}}_2^+, Q(2)) = \mathrm{Ind}(\not{\mathcal{D}}_2^+, P_2) + [\mathcal{P}_2 - Q(2)].$$

The Proposition is then an immediate consequence of the definition of the higher spectral flow and of the three previous equalities.

Remark. One can immediately extend the previous Proposition to the odd dimensional case; in such a case we need to fix Cl(1)-spectral sections.

Notation. In order to treat simultaneously the odd and even dimensional case, from now on we shall not write the superscript $+$ in the even dimensional case (see formula (22); thus both in the even and in the odd-dimensional case we shall simply write $\mathrm{Ind}^{APS}(\not{\mathcal{D}}, \mathcal{P})$ for the index class associated to $\not{\mathcal{D}}$ and the choice of a spectral section for the boundary operator. We shall implicitly choose Cl(1)-spectral sections in the odd-dimensional case.

Now, for two SK-equivalent pairs (M_1, r_1), (M_2, r_2) as in Definition 10, we can prove a *defect formula* for the difference $\mathrm{Ind}\,\mathcal{D}_{M_2}^{\mathrm{sign},r_2} - \mathrm{Ind}\,\mathcal{D}_{M_1}^{\mathrm{sign},r_1} \in K_*(C_r^*(\Gamma))$. First we perturb the metrics g_1 and g_2 so as to be product-like near F. Thanks to the additivity formula proved in Section 4 we have

$$\mathrm{Ind}(\mathcal{D}_{M_1}^{\mathrm{sign},r_1}) = \mathrm{Ind}^{APS}(\mathcal{D}_{M_+}^{\mathrm{sign},r_1}, \mathcal{P}_1) + \mathrm{Ind}^{APS}(\mathcal{D}_{M_-}^{\mathrm{sign},r_1}, 1 - \mathcal{P}_1) \tag{23}$$

with $\mathcal{P}_1$ a noncommutative spectral section for the boundary operator $\mathcal{D}_{\partial M_+}^{\mathrm{sign},r_1}$. Similarly,

$$\mathrm{Ind}(\mathcal{D}_{M_2}^{\mathrm{sign},r_2}) = \mathrm{Ind}^{APS}(\mathcal{D}_{M_+}^{\mathrm{sign},r_2}, \mathcal{P}_2) + \mathrm{Ind}^{APS}(\mathcal{D}_{M_-}^{\mathrm{sign},r_2}, 1 - \mathcal{P}_2) \tag{24}$$

with $\mathcal{P}_2$ a noncommutative spectral section for $\mathcal{D}_{\partial M_+}^{\mathrm{sign},r_2}$. Consider $(g_1)_{|M_+}$ and $(g_2)_{|M_+}$ and let $g_{+,u}$, with $u \in [1,2]$ a path of riemannian metrics on M_+ connecting them. We can choose these metrics to be product-like near the boundary of M_+. By assumption there is a continuous family of classifying maps $r_{+,u} : M_+ \to B\Gamma$, $u \in [1,2]$, connecting $(r_1)_{|M_+}$ and $(r_2)_{|M_+}$. Thus, there is a 1-parameter family of Dirac-type operators on M_+, $\{\mathcal{D}_{M_+}^{\mathrm{sign},u}\}_{u \in [1,2]}$ with boundary family $\{\mathcal{D}_{\partial M_+}^{\mathrm{sign},u}\}_{u \in [1,2]}$. We can thus apply the previous Proposition. We obtain

$$\mathrm{Ind}^{APS}(\mathcal{D}_{M_+}^{\mathrm{sign},r_2}, \mathcal{P}_2) - \mathrm{Ind}^{APS}(\mathcal{D}_{M_+}^{\mathrm{sign},r_1}, \mathcal{P}_1) = \mathrm{sf}(\{\mathcal{D}_{\partial M_+}^{\mathrm{sign},u}\}; \mathcal{P}_2, \mathcal{P}_1) \quad \text{in} \quad K_*(C_r^*(\Gamma)) \tag{25}$$

Notice that it does not make sense to take a path joining g_1 and g_2 (indeed, M_1 and M_2 are in general different manifolds).

We apply the same reasoning to M_-. Thus by assumption there is a continuous family of classifying maps $r_{-,u} : M_- \to B\Gamma$, $u \in [1,2]$, connecting $(r_1)_{|M_-}$ and $(r_2)_{|M_-}$ and, we choose a path of metrics $g_{-,u}, u \in [1,2]$, connecting $(g_1)_{|M_-}$ and $(g_2)_{|M_-}$. We obtain a family of operators $\mathcal{D}_{M_-}^{\mathrm{sign},u}$ connecting $\mathcal{D}_{M_-}^{\mathrm{sign},r_2}$ and $\mathcal{D}_{M_-}^{\mathrm{sign},r_1}$ together with the formula

$$\mathrm{Ind}^{APS}(\mathcal{D}_{M_-}^{\mathrm{sign},r_2}, 1 - \mathcal{P}_2) - \mathrm{Ind}^{APS}(\mathcal{D}_{M_-}^{\mathrm{sign},r_1}, 1 - \mathcal{P}_1) = \mathrm{sf}(\{\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\}; 1 - \mathcal{P}_2, 1 - \mathcal{P}_1) \tag{26}$$

Summarizing, in $K_*(C_r^*(\Gamma))$:

(27)
$$\operatorname{Ind}\mathcal{D}_{M_2}^{\mathrm{sign},r_2} - \operatorname{Ind}\mathcal{D}_{M_1}^{\mathrm{sign},r_1} = \mathrm{sf}(\{\mathcal{D}_{\partial M_+}^{\mathrm{sign},u}\};\mathcal{P}_2,\mathcal{P}_1) + \mathrm{sf}(\{\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\};1-\mathcal{P}_2,1-\mathcal{P}_1)$$

Notice that
$$\mathcal{D}_{\partial M_+}^{\mathrm{sign},1} = -\mathcal{D}_{\partial M_-}^{\mathrm{sign},1}, \quad \mathcal{D}_{\partial M_+}^{\mathrm{sign},2} = -\mathcal{D}_{\partial M_-}^{\mathrm{sign},2}$$
but that, in general, $\mathcal{D}_{\partial M_+}^{\mathrm{sign},u} \neq -\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}$ for $u \in (1,2)$.

Let us consider the S^1 (i.e. periodic) family $\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}$ obtained taking the first family from 1 to 2 and the opposite of the second family from 2 to 1. Thus, with an obvious and harmless abuse of notation,

$$\{\mathcal{D}_F(\theta)\}_{\theta\in S^1} := \{\mathcal{D}_{\partial M_+}^{\mathrm{sign},u}\}_{u\in[1,2]} \cup \{-\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\}_{u\in[2,1]}. \tag{28}$$

We shall use the fact that, in general, the spectral flow is additive with respect to the composition of path; this means that if $\{\mathcal{A}(t)\}_{t\in[a,b]}$ is a 1-parameter family of Dirac-type operators as in Definitions 5 or 9 and if $\{\mathcal{B}(t)\}_{t\in[b,c]}$ is a second such family such that $\mathcal{B}(b)=\mathcal{A}(b)$, then, for the family $\{\mathcal{D}(t)\}_{t\in[a,c]} := \{\mathcal{A}(t)\}\cup\{\mathcal{B}(t)\}$ the following additivity-formula holds

$$\mathrm{sf}(\{\mathcal{D}(t)\};\mathcal{P}_a,\mathcal{P}_c) = \mathrm{sf}(\{\mathcal{A}(t)\};\mathcal{P}_a,\mathcal{P}_b) + \mathrm{sf}(\{\mathcal{B}(t)\};\mathcal{P}_b,\mathcal{P}_c) \quad \text{in} \quad K_*(C_r^*(\Gamma)) \tag{29}$$

This formula follows immediately from the definition. Since
$$\mathrm{sf}(\{\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\}_{u\in[1,2]};1-\mathcal{P}_2,1-\mathcal{P}_1) = \mathrm{sf}(\{-\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\}_{u\in[2,1]};\mathcal{P}_1,\mathcal{P}_2)$$
we can then write

$$\mathrm{sf}(\{\mathcal{D}_{\partial M_+}^{\mathrm{sign},u}\};\mathcal{P}_2,\mathcal{P}_1) + \mathrm{sf}(\{\mathcal{D}_{\partial M_-}^{\mathrm{sign},u}\};1-\mathcal{P}_2,1-\mathcal{P}_1) = \mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}) \tag{30}$$

obtaining, finally, a proof of the following

Theorem 10. *The following formula holds in $K_*(C_r^*(\Gamma))$:*

$$\operatorname{Ind}\mathcal{D}_{M_2}^{\mathrm{sign},r_2} - \operatorname{Ind}\mathcal{D}_{M_1}^{\mathrm{sign},r_1} = \mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}) \tag{31}$$

where $\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}$ is the periodic family of twisted signatures operators on F defined by (28).

One important consequence of our discussion so far is the following

Corollary 2. *Assume that Γ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let (M_1,r_1), (M_2,r_2) be two SK-equivalent pairs as in Definition 10. If, with the above notations,*
$$\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}) = 0 \quad \text{in} \quad K_*(C_r^*(\Gamma))\otimes_{\mathbb{Z}}\mathbb{Q}$$
then the higher signatures of (M_1,r_1) and (M_2,r_2) are the same.

Proof. It is proven in [16] (just after Theorem 0.5) that, under the stated assumption on Γ, the equality of the two index classes imply the equality of the higher signatures. The Proposition is then a consequence of Theorem 10.

5.2. **Remarks.**

1. Formula 31 is a higher version of formula (25.22) in [5]. In the numerical case treated there, the *numerical* spectral flow is also equal to the *numerical* index of the mapping torus defined by the diffeomorphism $\psi = (\phi_2)^{-1} \circ \phi_1$. We shall now discuss the extension of the latter result to the higher context. We consider two SK-equivalent pairs (M_1, r_1), (M_2, r_2) as in Definition 10. We consider the mapping torus $M(F, \psi)$, which is obtained from $F \times [1,2] \sqcup F \times [1,2]$ by identifying $(\phi_1(x), 1)$ (in the first $F \times [1,2]$) with $(x, 1)$ (in the second $F \times [1,2]$) and similarly $(\phi_2(x), 2)$ with $(x, 2)$, for any $x \in F$. As explained in [14] (page 16), the mapping torus $M(F, \psi)$ comes with a map $\hat{r} : M(F, \psi) \to B\Gamma$ such that

$$(M_2, r_2) - (M_1, r_1) \text{ is bordant to } (M(F, \psi), \hat{r}) \tag{32}$$

Let us recall the argument: one considers the space Y obtained from $M_1 \times [0,3] \sqcup M_2 \times [0,3]$ and the following identifications: for $x \in F$ identify $(x, t) \in \partial M_+ \times [0,1]$ with $(\phi_1(x), t) \in \partial M_\times [0,1]$ and $(x, t) \in \partial M_+ \times [2,3]$ with $(\phi_2(x), t) \in \partial M_- \times [2,3]$. Then, after smoothing, $\partial Y = M_1 - M_2 - M(F, \psi)$; moreover the two homotopies $(r_j)_{|M_+} \simeq (r_j)_{|M_-}$ can be used to define a map $R : Y \to B\Gamma$, with $R_{|M_j} = r_j$; one then sets $\hat{r} := R_{|M(F,\psi)}$ which proves (32). Using the metrics considered in the discussion preceeding the statement of Theorem 10 we can endow the mapping torus $M(F, \psi)$ with a riemannian metric; we shall then denote by $\mathcal{D}^{sign,\hat{r}}_{M(F,\psi)}$ the signature operator with coefficients in the flat bundle associated with $\hat{r}$.

Proposition 9. *One has:*

$$\operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_2}_{M_2} - \operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_1}_{M_1} = \operatorname{Ind} \mathcal{D}^{sign,\hat{r}}_{M(F,\psi)}. \tag{33}$$

Proof. According to (32) the difference of pairs $[M_1, r_1] - [M_2, r_2]$ is bordant to $[M(F, \psi), \hat{r}]$. By the cobordism invariance of the index class (see [18], Proposition 2.3), one gets immediately the Proposition.

From the previous Proposition and Theorem 10 we immediately obtain the following higher analogue of Proposition 25.1 in [5]

Corollary 3. *With the notation introduced above the following formula holds in* $K_*(C^*_r(\Gamma))$

$$\operatorname{Ind} \mathcal{D}^{sign,\hat{r}}_{M(F,\psi)} = \mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta \in S^1}) \tag{34}$$

2. We have obtained formula (34) by exploiting two different ways for computing the difference $\operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_2}_{M_2} - \operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_1}_{M_1}$. In fact, a *direct* proof of this fact is essentially given in [32] using the intersection product in KK-theory. The results of Wu apply to any periodic family giving, at the K-theoretic level, the usual relationship between the spectral flow and the index.

3. Let $\mathcal{B}^\infty_\Gamma$ be the Connes-Moscovici algebra associated to Γ. Then, as a consequence of (34) we have that

$$\mathrm{Ch}\,(\operatorname{Ind} \mathcal{D}^{sign,\hat{r}}_{M(F,\psi)}) = \mathrm{Ch}\,(\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta \in S^1})) \tag{35}$$

holds in the topological non commutative de Rham homology $\overline{H}_*(\mathcal{B}^\infty_\Gamma)$ (see [13]) This formula, a weaker version of (34), can be proved *directly* without using the unpublished paper of Wu and using instead results of [10] (Theorem 3.2) together with their noncommutative generalizations in [18] (Theorem 5.3). We omit the details.

4. For future applications, we briefly point out that the defect formula (31) is still valid if one replaces the signature operators by Dirac type operators. In this generality one needs to give the appropiate definitions. Thus, we assume that we are given metrics g_1 on M_1 and g_2 on M_2 and Clifford bundles E_1 on M_1 and E_2 on M_2. We let D_1 and D_2 be two Dirac type operators on these bundles. We assume that the two bundles are obtained by a clutching construction involving two Clifford bundles $E_+ \to M_+$, $E_- \to M_-$ and two bundles isomorphisms $(E_+)_{|\partial M_+} \simeq (E_-)_{|\partial M_-}$ covering respectively ϕ_1 and ϕ_2. Given two classifying maps $r_j : M_j \to B\Gamma$ $(1 \leq j \leq 2)$ as above one defines two $\Lambda-$linear Dirac type operators $\mathcal{D}^{r_j}_{M_j}$. Moreover, assuming that the two paths of metrics $\{g_{+,u},\, u \in [1,2]\,\}$ and $\{g_{-,u},\, u \in [1,2]\,\}$ considered in Section 5.1 induce (now) a Clifford action on respectively E_+ and E_- for each $u \in [1,2]$, one may define an associated S^1-family $(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1})$ so that the defect formula (31) holds verbatim.

A typical example of this situation is given when M_j are spin manifolds and the operators are Dirac operators. We leave the details to the interested reader.

5.3. **Vanishing spectral flow in the even-dimensional case.**

In this subsection we consider two SK-equivalent pairs (M_1, r_1), (M_2, r_2) as in Definition 10 such that $\dim M_1 = \dim M_2 = 2m$. We shall give sufficient conditions for the noncommutative spectral flow appearing in the defect formula of Theorem 10 to be zero. Let F be as in the previous sub-section and consider $r = (r_1)_{|F} : F \to B\Gamma$.

Following [23] and then [15] we shall make the following:

Assumption A. *Let $\Omega^\ell_{(2)}(F, \mathcal{V}_r)$ denote the L^2_Λ-completion of $\Omega^\ell(F, \mathcal{V}_r)$. We assume that the operator*

$$d : \Omega^{m-1}_{(2)}(F, \mathcal{V}_r) \to \Omega^m_{(2)}(F, \mathcal{V}_r),$$

with domain equal to Λ-Sobolev space H^1_Λ, has closed image.

This assumption appeared for the first time in [23] (see Section 7); there it was assumed that the L^2-spectrum of the differential form laplacian of $\widetilde{F}$ (the covering defined by $r : F \to B\Gamma$), acting on the vector space of differential forms of degree m, has a full gap at zero. Assumption A (which is a homotopy-invariant assumption) is equivalent to the slightly weaker assumption that the differential form laplacian of $\widetilde{F}$ has a strictly positive spectrum on $\Omega^{m-1}(\widetilde{F})/\ker(d)$. We refer the reader to [15] for a proof of this fact.

We first recall (and slightly extend) the definition of symmetric spectral section [19]. Under Assumption A we have

$$\Omega^m_{(2)}(F, \mathcal{V}_r) = d\Omega^{m-1}_{(2)}(F, \mathcal{V}_r) \oplus (d\Omega^{m-1}_{(2)}(F, \mathcal{V}_r))^\perp$$

$$\Omega^{m-1}_{(2)}(F, \mathcal{V}_r) = d^*\Omega^m_{(2)}(F, \mathcal{V}_r) \oplus (d^*\Omega^m_{(2)}(F, \mathcal{V}_r))^\perp.$$

Then, we can write $\Omega^*(F, \mathcal{V}_r) = V \oplus W$ where (we abbreviate $\Omega^*_{\mathcal{V}} = \Omega^*(F, \mathcal{V}_r)$):

$$V = d^*\Omega^m_{\mathcal{V}} + d\Omega^{m-1}_{\mathcal{V}}, \quad W = \Omega^{<}_{\mathcal{V}} \oplus \Omega^{>}_{\mathcal{V}}$$

$$\Omega^{<}_{\mathcal{V}} = \Omega^0_{\mathcal{V}} \oplus \ldots \oplus \Omega^{m-2}_{\mathcal{V}} \oplus (d^*\Omega^m_{\mathcal{V}})^\perp,$$

$$\Omega^{>}_{\mathcal{V}} = (d\Omega^{m-1}_{\mathcal{V}})^\perp \oplus \Omega^{m+1}_{\mathcal{V}} \oplus \ldots \oplus \Omega^{2m-1}_{\mathcal{V}}.$$

It is clear that $\mathcal{D}^{sign,r}_F$ sends V (resp. W) into itself. Using Assumption A and proceeding as in the proof of Lemma 2.1 of [15], one checks easily that $\mathcal{D}^{sign,r}_F$

induces an invertible operator on the $L^2_\Lambda-$completion of V (with domain H^1) and we denote by $\Pi_>$ the projection onto the positive part. Moreover, the proof of Theorem 2 shows immediately that the restriction of $\mathcal{D}_F^{sign,r}$ to the $L^2_\Lambda-$completion of W admits a spectral section. Notice that although W is not the space of sections of a $\Lambda-$bundle over F the proof of Theorem 2 still applies in this case. Then, Lemma 4.3 of [19] shows that $\mathcal{D}_F^{sign,r}$ admits a symmetric spectral section $\mathcal{P}$ in the sense that $\mathcal{P}$ is diagonal with respect to the decomposition $\Omega^*(F,\mathcal{V}_r) = V \oplus W$ and:

$$\mathcal{P}_{|V} = \Pi_>, \quad \mathcal{P}_{|W} \circ \alpha + \alpha \circ \mathcal{P}_{|W} = \alpha$$

where α is the involution of W equal to identity on $\Omega^<_{\mathcal{V}}$ and to minus the identity on $\Omega^>_{\mathcal{V}}$. Recall from [19] Prop. 4.4 that if $\mathcal{Q}$ is another symmetric spectral section then $[\mathcal{P} - \mathcal{Q}] = 0$ in $K_0(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$. We immediately obtain the following

Proposition 10. *Let (M_1,r_1), (M_2,r_2) be two SK-equivalent pairs as in Definition 10. We set $r := (r_1)_{|F}$. If Assumption A holds for (F,r) then*

$$\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}) = 0 \quad in \quad K_*(C^*_r(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. In the definition of $\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1})$, we can assume that *all* the spectral sections involved are symmetric. Since, by definition, the noncommutative spectral flow is given in terms of difference classes associated to these symmetric spectral sections, it follows by Proposition 4.4 in [19] that $\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1})$ is zero in $K_0(C^*_r(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Proposition is proved.

Remark. Notice that because of the homotopy-invariance of Assumption A we could equivalently set $r := (r_2)_{|F}$.

We have now obtained an *analytic* proof of the main result in [16]

Theorem 11. *Assume that Γ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let (M_1,r_1) and (M_2,r_2) be two SK-equivalent pairs as in Definition 10. We assume that Assumption A holds for (F,r) with $r := (r_1)_{|F}$. Then the higher signatures of (M_1,r_1) and (M_2,r_2) are the same.*

The proof is a direct consequence of Proposition 10 and of Corollary 2.

Remark. The previous Theorem was first proved in [15] in the case of virtually nilpotent or Gromov Hyperbolic groups. It was then extended by Leichtnam-Lueck ([16]), using techniques from algebraic surgery, to the case where Γ satisfies the assumption of Theorem 11.

5.4. **Vanishing spectral flow in the odd dimensional case.**

In this subsection we consider two SK-equivalent pairs (M_1,r_1), (M_2,r_2) as in Definition 10 such that $\dim M_1 = \dim M_2 = 2m+1$. Let F be as in the previous section and consider $r = (r_1)_{|F} : F \to B\Gamma$.

As in the even case we shall make the following assumption on (F,r):

Assumption B. *The operator*

$$d : \Omega^{m-1}_{(2)}(F,\mathcal{V}_r) \to \Omega^m_{(2)}(F,\mathcal{V}_r)$$

with domain equal to Λ-Sobolev space H^1_Λ, has closed range.

It is proved in [15] that Assumption B is equivalent to the hypothesis that L^2-spectrum of the differential form laplacian of $\widetilde{F}$ acting on the vector space of differential forms of degree m admits a gap at zero.

Assumption B (in the slightly stronger form of requiring L^2-invertibility in middle degree) appears for the first time in Section 7 of [23].

Now we recall results and notations allowing to introduce the concept of symmetric Cl(1)−spectral section. We define the rescaled flat exterior derivative $\tilde{d}$: $\Omega^*(F,\mathcal{V}_r) \to \Omega^{*+1}(F,\mathcal{V}_r)$ by setting:

$$\forall \omega \in \Omega^{|\omega|}(F,\mathcal{V}_r), \quad \tilde{d}(\omega) = i^{|\omega|} d\omega.$$

Then using the Hodge duality operator $*$ one defines the involution τ:

$$\forall \omega \in \Omega^{|\omega|}(F,\mathcal{V}_r), \quad \tau(\omega) = i^{-|\omega|(2m-|\omega|)} * \omega.$$

We refer to the Section 1 of [15] for the definition of the quadratic form Q and the $\Lambda-$hermitian scalar product $<;>$ on $\Omega^*(F,\mathcal{V}_r)$. The signature operator $\mathcal{D}_F^{\mathrm{sign},r}$ is then equal to $\tilde{d} - \tau\tilde{d}\tau$.

Let $\mathrm{H}^m(F,\mathcal{V}_r)$ denote the $m-$th cohomology group of the locally constant sheaf $\mathcal{V}_r$. Using Hodge duality and Assumption B one proves easily that the two following orthogonal decomposition holds:

$$\Omega^m_{(2)}(F,\mathcal{V}_r) = \mathrm{H}^m(F,\mathcal{V}_r) \oplus \overline{\tau\tilde{d}\tau(\Omega^{m+1}(F,\mathcal{V}_r))} \oplus \overline{\tilde{d}(\Omega^{m-1}(F,\mathcal{V}_r))}.$$

$$\Omega^*_{(2)}(F,\mathcal{V}_r) = V \oplus W \quad \text{with}$$
$$V = \Omega^m_{(2)}(F,\mathcal{V}_r) \oplus \mathcal{D}_F^{\mathrm{sign},r}(H^1_\Lambda(F;\wedge^m T^*F \otimes \mathcal{V}_r))\,, \quad W = V^\perp$$

V being a closed complementable $\Lambda-$Hilbert submodule of $\Omega^*_{(2)}(F,\mathcal{V}_r)$. We observe that both $\mathcal{D}_F^{\mathrm{sign},r}$ and the grading τ send V (resp. W) into itself.

We shall consider the involution α of W such that α induces Id (resp. $-$Id) on the differential forms of degree $< m$ (resp. $> m$).

Proposition 11. *At the expense of replacing F by two disjoint copies of F, one can find a real number $R > 0$ such that for any spectral section P' of $(\mathcal{D}_F^{\mathrm{sign},r})_{|V}$ satisfying $P' \circ \xi(\mathcal{D}_F^{\mathrm{sign},r}) = 0$ for any $\xi \in C^\infty(\mathbb{R},\mathbb{R})$ vanishing on $[R,+\infty[$, $\ker(P' + \tau P'\tau)$ is a finitely generated projective Λ-submodule N of V such that:*
0) $\mathcal{D}_F^{\mathrm{sign},r}(N) \subset N$, $N^+ \simeq N^-$.
1) $\tau(N) \subset N$ and $\mathrm{H}^m(F,\mathcal{V}_r) \subset N$ as a Λ-submodule.
2) N admits a lagrangian Λ-submodule L with respect to the quadratic form Q and the orthogonal $\Lambda-$projection P_L from N onto L is well defined.
3) $V = N \oplus N^\perp$.
We will say that such an L is a stable lagrangian of $\mathrm{H}^m(F,\mathcal{V}_r)$ (associated with N).

Proof. The index class of the signature operator on F with values in $\mathcal{V}_r$ is zero in $K_0(\Lambda)$, by cobordism invariance. Notice that this index class is the sum of the index classes associated to the restrictions $\mathcal{D}_V$, $\mathcal{D}_W$ of the signature operator $\mathcal{D}_F^{\mathrm{sign},r}$ to V and W respectively. Under Assumption B, one checks easily that

$$\alpha(\mathcal{D}_W^+)^*\alpha = \mathcal{D}_W^+$$

and so $2\,\mathrm{Ind}\,\mathcal{D}_W^+ = 0$ in $K_0(\Lambda)$. Replacing F by the disjoint union of two copies of F, we may assume that $\mathrm{Ind}\,\mathcal{D}_W^+ = 0$ in $K_0(\Lambda)$. Therefore, we may conclude that the index class associated to $\mathcal{D}_F^{\mathrm{sign},r}$ restricted to V is also zero. Extending the proof of Proposition 2 page 295 of [28] to the case of $(\mathcal{D}_F^{\mathrm{sign},r})_{|V}$ along the lines of [19] (see Lemma 4.3), we infer that there exists a real number $R > 0$ such that for any spectral section P' as in the statement of the Theorem, $N = \ker(P' + \tau P' \tau)$ is a finitely generated projective submodule of V containing $\mathrm{H}^m(F, \mathcal{V}_r)$ and such that:

a) N is the range of a projection of V and thus $N \oplus N^\perp = V$.

b) $N = N^+ \oplus N^-$, with $N^\pm = \{\omega\, ;\, \tau\omega = \pm\omega\}$ and N^+ isomorphic to N^-.

Let then $j : N^+ \to N^-$ be a unitary isomorphism; our lagrangian L is given by

$$\{\omega + j(\omega)\ , \omega \in N^+\}\,.$$

The Proposition is proved.

Remarks.

1. This notion of stable lagrangian is essentially the same as the one used in Section 3 of [15].

2. In the previous Proposition one may assume that N is included in the range of $\psi(\mathcal{D}_F^{\mathrm{sign},r}{}_{|V})$ for a suitable function $\psi \in C^\infty_{comp}(\mathbb{R};\mathbb{R})$.

Now, let L be a stable lagrangian of $\mathrm{H}^m(F, \mathcal{V}_r)$ (associated with N) as in Proposition 11. We observe that $\mathcal{D}_F^{\mathrm{sign},r}$ is diagonal with respect to the decomposition

$$\Omega^*_{(2)}(F, \mathcal{V}_r) = (N \oplus N^\perp) \oplus W$$

and that the restriction of $\mathcal{D}_F^{\mathrm{sign},r}$ to $N^\perp$ is invertible. We shall denote by $\Pi_>(N^\perp)$ the APS projection onto the positive part of the restriction of $\mathcal{D}_F^{\mathrm{sign},r}$ to $N^\perp$. We then may give the following:

Definition 11. Let R be a spectral section $\in B_\Lambda(W)$ for $\mathcal{D}_F^{\mathrm{sign},r}{}_{|W}$ which is Cl(1) for both α and τ. The self-adjoint projection

$$\mathcal{P}(L, R) = P_L \oplus \Pi_>(N^\perp) \oplus R$$

of $B_\Lambda(\Omega^*_{(2)}(F, \mathcal{V}_r))$ is called symmetric Cl(1)$-$spectral section for $\mathcal{D}_F^{\mathrm{sign},r}$.

One checks the existence of a spectral section R as in the above Proposition by proceeding as in the proof of Lemma 4.3 of [19]. Therefore, there exists such symmetric Cl(1)$-$spectral sections.

Let L and L' be two stable lagrangians for $\mathrm{H}^m(F, \mathcal{V}_r)$, as in Proposition 11. The proof of the Proposition and the arguments of [28] (page 295) shows that if in the statement of Proposition 11 one replaces the real number R by a bigger one then $N^\pm$ gets replaced by $N^\pm \oplus M^\pm$ where $(\mathcal{D}_F^{\mathrm{sign},r})^+$ induces an isomorphism from the finitely generated projective $C^*_r(\Gamma)-$module M^+ onto the $C^*_r(\Gamma)-$module M^-. Thus, at the expense of replacing L by

$$\{\, \omega \oplus \beta + j(\omega) \oplus (\mathcal{D}_F^{\mathrm{sign},r})^+(\beta),\ \ (\omega, \beta) \in N^+ \times M^+\,\}$$

(and similarly for L'), one may assume that both L and L' are associated with the same N.

Definition 12. Let L, L' be two lagrangians as above. Let P_L, $P_{L'}$ be the Λ-endomorphisms of N given by the orthogonal projections onto L, L' respectively. According to Definition 8 these two projections define a difference class $[P_L - P_{L'}] \in K_0(C^0_c(]0,\pi[) \otimes \Lambda) = K_1(\Lambda)$; we set $[L - L'] := [P_L - P_{L'}]$

Remarks.

1. The class $[L - L'] \in K_1(\Lambda)$ depends only on L and L'.

2. One can give a different definition of the class $[L - L'] \in K_1(\Lambda)$ (see [15] Section 3): if j, j' are two unitary isomorphims $N^+ \to N^-$ such that $L = \{\omega + j(\omega)/\omega \in N^+\}$, $L' = \{\omega + j'(\omega)/\omega \in N^+\}$, one defines the class $[L - L']$ to be $[j \circ (j')^{-1}] \in K_1(\Lambda)$. The two definitions should be compatible through the suspension isomorphism, but we have not looked into the details.

We have the following *relative* result:

Proposition 12. *Let L and L' be two stable lagrangians as in Definition 12. Let $\mathcal{P}(L, R)$ and $\mathcal{P}(L', R')$ be two symmetric* $\mathrm{Cl}(1)-$*spectral sections for $\mathcal{D}_F^{\mathrm{sign},r}$ as in Definition 11, then one has*

$$[\mathcal{P}(L, R) - \mathcal{P}(L', R')] = [L - L'] \quad \text{in} \quad K_1(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. We set $P_1 = \mathcal{P}(L, R)$ and $P_2 = \mathcal{P}(L', R')$ and assume that L and L' are associated with (the same) N. We then consider the odd self-adjoint $\Lambda-$compact operator $\mathcal{A}^0_{P_j}$ associated with P_j $(j \in \{1, 2\})$ as in Definition 8. Thus $\mathcal{D}_F^{\mathrm{sign},r} + \mathcal{A}^0_{P_j}$ is invertible and we can moreover assume that $\mathcal{A}^0_{P_j}$ is diagonal with respect to the decomposition

$$\Omega^*_{(2)}(F, \mathcal{V}_r) = (N \oplus N^\perp) \oplus W$$

and that its restriction to W, denoted $\mathcal{A}^0_j(W)$, is odd with respect to the grading α. Of course we can assume that the restrictions of $\mathcal{A}^0_{P_j}$ to $N^\perp$ coincide.

Now, for $j \in \{1, 2\}$ and $t \in [0, \pi]$, we denote by $P_j(t)$ the projection onto the positive part of

$$\tau(\,\mathrm{Id} + [{\mathcal{D}_F^{\mathrm{sign},r}}_{|W} + \mathcal{A}^0_j(W)]^2\,)^{\frac{1}{2}} \cos t + ({\mathcal{D}_F^{\mathrm{sign},r}}_{|W} + \mathcal{A}^0_j(W)) \sin t.$$

Since $\tau\alpha = -\alpha\tau$ and both ${\mathcal{D}_F^{\mathrm{sign},r}}_{|W}$ and the $\mathcal{A}^0_j(W)$ are odd with respect to the grading α, one checks immediately that $\alpha(P_1(t) - P_2(t)\alpha = -(P_1(t) - P_2(t)$ for any $t \in [0, \pi]$. Therefore

$$[(P_1(t) - P_2(t))_{t\in[0,\pi]}] = -[(P_2(t) - P_1(t))_{t\in[0,\pi]}] \text{ in } K_0(C^0_c(]0, \pi[) \otimes \Lambda) \tag{36}$$

and this $K-$theory class is rationnally zero.

Next, we denote by $\mathcal{A}^0_j(N)$ the restriction of $\mathcal{A}^0_{P_j}$ to N for $j \in \{1, 2\}$. For $t \in [0, \pi]$ we denote by $Q_j(t)$ the projection onto the positive part of

$$\tau(\,\mathrm{Id} + [{\mathcal{D}_F^{\mathrm{sign},r}}_{|N} + \mathcal{A}^0_j(N)]^2\,)^{\frac{1}{2}} \cos t + ({\mathcal{D}_F^{\mathrm{sign},r}}_{|N} + \mathcal{A}^0_j(N)) \sin t.$$

Then by Definition 12 of the difference class one has:

$$[(Q_1(t) - Q_2(t))_{t\in[0,\pi]}] = [L - L'] \text{ in } K_0(C^0_c(]0, \pi[) \otimes \Lambda) \simeq K_1(\Lambda) \tag{37}$$

By combining the two equations (36) and (37) and recalling Definition 8, one gets immediately the result of the Proposition.

We recall that (M_1, r_1) and (M_2, r_2) are two SK-equivalent pairs of dimension $2m+1$ as in Definition 10. We consider the periodic family $(\mathcal{D}_F(\theta))_{\theta\in[0,2\pi]}$ (defined in (28)). By following the proof of Proposition 11 we may construct two continuous families of spaces N_θ and V_θ, each of them being associated with $\mathcal{D}_F(\theta)$, so that $(N_\theta)_{\theta\in[0,2\pi]}$ defines a bundle over $[0, 2\pi]$ of finitely generated projective $C^*_r(\Gamma)-$module with $N_0 = N_{2\pi}$

Let $(L_\theta)_{\theta\in[0,2\pi]}$ be a continuous family of stable lagrangians of $(\mathcal{D}_F(\theta))_{\theta\in[0,2\pi]}$; there exists a total spectral section $(P_\theta)_{\theta\in[0,2\pi]}$ for the periodic family $(\mathcal{D}_F(\theta))_{\theta\in[0,2\pi]}$

where, for each $\theta \in [0, 2\pi]$, P_θ can be chosen to be a $\mathrm{Cl}(1)-$*symmetric* spectral section of the form:

$$P_\theta = \mathcal{P}(L_\theta, R_\theta)$$

with the spectral sections R_θ varying continuously with respect to $\theta \in [0, 2\pi]$. By the definition of noncommutative spectral flow of a periodic family and by Proposition 12 we obtain:

$$\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta \in S^1}) := [\mathcal{P}(L_{2\pi}, R_{2\pi}) - \mathcal{P}(L_0, R_0)] = [L_{2\pi} - L_0]$$

Summarizing for a 1-parameter family of stable lagrangians as above:

$$\operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_2}_{M_2} - \operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_1}_{M_1} = [L_{2\pi} - L_0] \tag{38}$$

Definition 13. We shall say that the above family of stable lagrangian is $K_1(C^*_r(\Gamma))$-periodic if $[L_{2\pi} - L_0] = 0$ in $K_1(C^*_r(\Gamma))$.

We can state, finally, the main result of this subsection:

Proposition 13. *Let (M_1, r_1) and (M_2, r_2) be two SK-equivalent pairs of dimension $2m+1$ as in Definition 10. If Assumption B holds for $(F, r = (r_1)_{|F})$ and if the family $(\mathcal{D}_F(\theta))_{\theta \in [0, 2\pi]}$ admits a $K_1(C^*_r(\Gamma))$-periodic family of stable lagrangians then*

$$\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta \in S^1}) = 0 \quad \text{in} \quad K_1(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

and so

$$\operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_2}_{M_2} = \operatorname{Ind} \mathcal{D}^{\mathrm{sign}, r_1}_{M_1} \quad \text{in} \quad K_1(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The previous Proposition implies the main result of [16]:

Theorem 12. *Assume that Γ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let (M_1, r_1) and (M_2, r_2) be two SK-equivalent pairs of dimension $2m+1$ as in Definition 10. If Assumption B holds for (F, r) and if the family $(\mathcal{D}_F(\theta))_{\theta \in [0, 2\pi]}$ admits a $K_1(C^*_r(\Gamma))$-periodic family of stable lagrangians then (M_1, r_1) and (M_2, r_2) have the same higher signatures.*

Remark. The previous Theorem was first proved in [15] in the case of virtually nilpotent of Gromov Hyperbolic groups. It was then extended by Leichtnam-Lueck ([16]), using techniques from algebraic surgery, to the case where Γ satisfies the condition of Theorem 12. (Notice that in [15] [16] the result is stated under the slightly stronger assumption that it should exists a ψ-*invariant* stable lagrangian, with $\psi = \phi_1 \circ \phi_2^{-1}$. However, Proposition 3.7 in [15] shows that this stronger assumption can be relaxed as in Theorem 12 above; the same is true for [16], using results from algebraic surgery about stable formations.)

References

[1] M. Atiyah, I. Singer. *Index theory of skew-adjoint Fredholm operators*, Inst. Hautes Etudes Sci. Publ. Math. **37**, 1969, pages 305-326.

[2] P. Baum, A. Connes. *Leafwise homotopy equivalence and rational Pontrjagin classes. Foliations*, Advances Studies in Pure Math., 5, North-Holland, Amsterdam, 1985, pages 1-14.

[3] B. Blackadar. *K-Theory of operator algebras*, Math. Sciences Research Institute Publications, 1998.

[4] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*, Springer Verlag 298, 1992.

[5] B. Booss-Bavnek, K. Wojciechowski *Elliptic boundary problems for Dirac operators*, Mathematics: Theory and Applications, Birkhauser 1993.

[6] U. Bunke. *On the gluing problem for the η−invariant*, Journal of Differential Geometry, **41**, 1995, pages 397-448.

[7] A. Connes. *Noncommutative Geometry.* Academic Press. San Diego.

[8] A. Connes, H. Moscovici. *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology. **29**, 1990, pages 345-388.

[9] X.Dai, W. Zhang. *Splitting of the family index*, Commun. Math. Phys. **182**, 1996, pages 303-318.

[10] X.Dai, W. Zhang. *Higher Spectral Flow*, Journal of Funct. Analysis, **157**, 1998, pages 432-469.

[11] N. Dencker. *On the propagation of polarization sets for systems of real principal type*, Journal of Funct. Analysis, **46**, No. 3, 1982, pages 351-372.

[12] M. Hilsum. *Index classes of Hilbert modules with boundary*, Preprint Paris 6, March 2001.

[13] M. Karoubi. *Homologie cyclique et K-Théorie*, Astérisque **149**, 1987.

[14] U. Karras, M. Kreck, W. Neumann and E. Ossa. *Cutting and pasting of manifolds. SK-groups*, Publish or Perish, Boston 1973.

[15] E. Leichtnam, J.Lott and P.Piazza. *On the homotopy invariance of higher signatures for manifolds with boundary*, Journal of Differential Geometry, **54**, 2000, pages 561-633.

[16] E. Leichtnam, W.Lueck. *On the cut and paste property of higher signatures of a closed oriented manifold*, Preprint 2000, to appear in Topology.

[17] E. Leichtnam, P. Piazza. *The b−pseudo-differential calculus on Galois coverings and a higher Atiyah-Patodi-Singer index theorem*, Mémoires de la Société Mathématiques de France **68**, 1997.

[18] E. Leichtnam, P. Piazza. *Spectral sections and higher Atiyah-Patodi-Singer index theory on Galois coverings*, GAFA, Vol. 8, 1998, pages 17-58.

[19] E. Leichtnam, P. Piazza. *A Higher Atiyah-Patodi-Singer Index theorem for the signature operator on Galois Coverings*, Annals of Global Analysis and Geometry, **18**, 2000, pages 171-189.

[20] E. Leichtnam, P. Piazza. *Homotopy invariance of twisted higher signatures on manifolds with boundary*, Bull. Soc. math. France, **127**, 1999, pages 307-331.

[21] E. Leichtnam, P. Piazza. *On higher eta invariants and metrics of positive scalar curvature.* Preprint June 2000.

[22] J. Lott. *Superconnections and higher index theory* GAFA, bf 2, pages. 421-454.

[23] J. Lott. *Higher eta invariants*, K-Theory, **6**, 1992, pages 191-233.

[24] J. Lott. *Signatures and higher signatures on S^1-quotients.* Math. Annalen **316**, 2000, pages 617-657.

[25] R. Mazzeo, P. Piazza. *Dirac operators, heat kernels and microlocal analysis. Part II: Analytic surgery.* Rendiconti di Matematica e delle sue applicazioni (Roma),**18** 1998.

[26] R. Melrose. *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, **4**, 1993, A K Peters.

[27] R. Melrose, P. Piazza. *An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary* , Journal of Differential Geometry, **46**, 1997, pages 287-334.

[28] R. Melrose, P. Piazza. *An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary* , Journal of Differential Geometry, **46**, 1997, pages 287-334.

[29] W. Neumann. *Manifold cutting and pasting groups.* Topology, **14**, 1975, page 237.

[30] P. Piazza. *Dirac operators, heat kernels and microlocal analysis. Part I: Family index theory.* Rendiconti del Circolo Matematico di Palermo, Supplemento, Serie II, **49** 1997.

[31] N. Wegge-Olsen. *$K-$Theory and C^*-algebras*, Oxford University Press, 1993.

[32] F. Wu. *The noncommutative spectral flow*, unpublished preprint, 1997.

[33] F. Wu. *The Higher $\Gamma-$index for coverings of manifolds with boundaries*, Fields Institute Communications, Volume **17**, 1997, pages 169-183.

Institut de Jussieu et CNRS, Etage 7E, 175 rue du Chevaleret, 75013, Paris, France
E-mail address: `leicht@math.jussieu.fr`

Dipartimento di Matematica G. Castelnuovo, Università di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Rome, Italy
E-mail address: `piazza@mat.uniroma1.it`