CERAMI (C) CONDITION AND MOUNTAIN PASS THEOREM FOR MULTIVALUED MAPPINGS

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1 Introduction

In the paper [11] is developed a critical point theory for multivalued mappings with closed graph. In this work we prove a general minmax result for multivalued mappings F with closed graph and as application we give an existence result of critical point of this mapping which satisfies the Cerami (C) condition.
Many papers has been devoted to prove the existence of the critical points provided that the respective functional satisfy the suitable Palais-Smale condition. In this work we can guarantee this fact assuming a weaker condition and in same time working with a very general class of functionals. From the viewpoint of the "mountain pass" it is more natural to work on the graph of the functional than on its domain, see [11].

In order to obtain this result, the paper is organized as follows. In Section 2 we recall the notions which are used in the paper and we compare the weak slopes introduced in [11] and [7]. In Section 3 we prove the *generalized Willem deformation lemma* for continuous functionals and we introduce the notion of homotopy stability with boundary in the $graph F$. We also state a localization theorem. In Section 4 we extend for multivalued mappings a result from [6] which states a relation between the weak slopes if we change the metric of the $graph F$. Finally, in Section 5 using the Cerami (C) conditon for our case, we state the main result of this paper, i.e. the mountain pass type result for multivalued mappings, which is an extension of the theorem of Ghoussoub-Preiss, see [8], [13] .

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2 Preliminaries

We recall some definitions and results from the papers [7], [3] and [11].

Definition 2.1 *Let (X,d) be a complete metric space and let* $f : X \to \mathbb{R}$ *be a continuous function and* $u \in X$ *a fixed element. We denote by* $|df|(u)$ *the supremum of the* $\sigma \in [0,\infty[$ *such that there exist* $\delta > 0$ *and a continuous map*

$$\mathcal{H} : B(u,\delta) \times [0,\delta] \to \mathbb{R}$$

such that $\forall v \in B(u,\delta)$ *for all* $t \in [0,\delta]$ *we have*

$$(a) \quad d(\mathcal{H}(v,t),v) \leq t$$

$$(b) \quad f(\mathcal{H}(v,t)) \leq f(v) - \sigma t$$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

Definition 2.2 *Let* $f : X \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function. We define the function*

$$\mathcal{G}_f : epi(f) \to \mathbb{R}$$

putting

$$epi(f) = \{(u,\xi) \in X \times \mathbb{R} : f(u) \leq \xi\} \quad and \quad \mathcal{G}_f(u,\xi) = \xi.$$

In the following $epi(f)$ *will be endowed with the metric*

$$d_{ep}((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi-\mu)^2)^{\frac{1}{2}}.$$

Of course $epi(f)$ is closed in $X \times \mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u,\xi) \leq 1$ for every $(u,\xi) \in epi(f)$.

Proposition 2.3([7, Proposition 2.3]) *Let* $f : X \to \mathbb{R}$ *be a continuous function and let* $(u,\xi) \in epi(f)$. *Then*

$$|d\mathcal{G}_f|(u,\xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1+|df|(u)^2}}, & if \;\; f(u) = \xi and \;\; |df|(u) < \infty, \\ 1, & if \;\; f(u) < \xi or \;\; |df|(u) = \infty. \end{cases}$$

Theorem 2.4([3, Theorem 2.11]) *Let* (X,d) *be a complete metric space and let* $f : X \longrightarrow \mathbb{R}$ *be a continuous function,* C *a closed subset of* X *and* $\delta, \sigma > 0$ *such that*

$$d(u,C) \leq \delta \Longrightarrow |df|(u) > \sigma.$$

Then there exists a continuous map $\eta : X \times [0,\delta] \to X$ *such that*

1) $d(\eta(u,t),u)) \leq t$,
2) $f(\eta(u,t)) \leq f(u)$,

3) $d(u, C) \geq \delta \Longrightarrow \eta(u,t) = u$,
4) $u \in C \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma t$.

In the next let (X, d) be a metric space and let $F : X \to \overline{\mathbb{R}}$ be a multivalued mapping with closed graph and nonempty values. We denote by

$$graphF = \{(u,c) \in X \times \mathbb{R} \ \mid \ c \in F(u) \ \}.$$

The set graph F is a metric space endowed with the metric

$$d_g((u,b),(v,c)) = \sqrt{d^2(u,v) + |b-c|^2}.$$

Now, we recall the definition of*weak slope* for F, see Frigon [11].

Definition 2.5 *Let $F : X \longrightarrow \overline{\mathbb{R}}$ be a multivalued mapping with closed graph and let $(u,b) \in graphF$ be a point. The* **weak slope** *of F at (u,b), denoted by $|dF|(u,b)$ is the supremum of $\sigma \in [0,\infty[$ such that there exists $\delta > 0$ and a continuous function*

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2); B((u,b),\delta) \times [0,\delta] \longrightarrow graphF,$$

(where $B((u,b),\delta)$ is the open ball in $graphF$ centered at (u,b) of radius δ) such that
(2.1) $\quad d_g(\mathcal{H}((v,c),t),(v,c)) \leq t\sqrt{1+\sigma^2}$;
(2.2) $\quad \mathcal{H}_2((v,c),t) \leq c - \sigma t$.

Proposition 2.6 Let $(u,b) \in graphF$. Then

$$|d\mathcal{G}_F|(u,b) \geq \begin{cases} \dfrac{|dF|(u,b)}{\sqrt{1+|dF|^2(u,b)}}, & |dF|(u,b) < \infty \\ 1\ , & |dF|(u,b) = \infty \end{cases}$$

Proof If $|dF|(u,b) = 0$, the relation is obvious. Otherwise, let $0 < \sigma < |dF|(u,b)$ and $H = (H_1, H_2) : B((u,b),\delta) \times [0,\delta] \to graphF$ such that
(2.3) $\quad d_g(H((v,c),t),(v,c)) \leq t\sqrt{1+\sigma^2}$
(2.4) $\quad H_2((v,c),t) \leq c - \sigma t$, if $(v,c) \in B((u,b),\delta) \times [0,\delta] \cap graphF$.
Let $\mathcal{H} : B((u,b),\delta) \times [0,\delta] \to graphF$ be a function defined by

$$\mathcal{H}((v,c),t) = H((v,c), \frac{t}{\sqrt{1+\sigma^2}}).$$

The function $\mathcal{H}$ is well defined. In fact $H((v,c), \frac{t}{\sqrt{1+\sigma^2}}) \in graphF$ and $0 \leq \dfrac{t}{\sqrt{1+\sigma^2}} \leq \delta$, for all $t \in [0,\delta]$. Of course $\mathcal{H}$ is continuous. From the relation (2.3) we get $d_g(\mathcal{H}((v,c),t),(v,c)) = d_g(H((v,c), \dfrac{t}{\sqrt{1+\sigma^2}}),(v,c)) \leq \dfrac{t}{\sqrt{1+\sigma^2}}\sqrt{1+\sigma^2} = t$.

Thus we have
(2.5) $\quad d_g(\mathcal{H}((v,c),t),(v,c)) \leq t$.

Using the relation (2.4) we get
$\mathcal{G}_F(\mathcal{H}((v,c),t)) = \mathcal{G}_F(H((v,c),\dfrac{t}{\sqrt{1+\sigma^2}})) = H_2((v,c),\dfrac{t}{\sqrt{1+\sigma^2}}) \le$
$c - \sigma\dfrac{t}{\sqrt{1+\sigma^2}} = \mathcal{G}_F(v,c) - \dfrac{\sigma}{\sqrt{1+\sigma^2}}t.$
Thus we have
(2.6) $\quad \mathcal{G}_F(\mathcal{H}((v,c),t) \le \mathcal{G}_F(v,c) - \dfrac{\sigma}{\sqrt{1+\sigma^2}}t.$
Using the relations (2.5) and (2.6) we have $|d\mathcal{G}_F|(u,b) \ge \dfrac{\sigma}{\sqrt{1+\sigma^2}}$.
From this relation we have

$$|d\mathcal{G}_F|(u,b) \ge \begin{cases} \dfrac{|dF|(u,b)}{\sqrt{1+|dF|^2(u,b)}}, & |dF|(u,b) < \infty \\ 1\ , & |dF|(u,b) = \infty \end{cases}$$

3 A minmax principle

In this section we prove the "Quantitative deformation lemma" of Willem for continuous functionals and a general minmax result which will be used in the last section. We use the following notations:

$$f^c = \{x \in X| \ \ f(x) \le c\}$$

$$f_c = \{x \in X| \ \ f(x) \ \ \ge c\}$$

$$C_\delta = \{x \in X| \ \ d(x,C) \le \delta\},\ \delta > 0$$

Theorem 3.1 *Let (X,d) be a complete metric space, $f : X \longrightarrow \mathbb{R}$ a continuous function, C a closed subset of X, $c \in \mathbb{R}$ and $\varepsilon, \lambda > 0$. Suppose that*
(3.0) $\quad C \cap f^{c+\varepsilon'} \cap f_{c-\varepsilon'} \ne \emptyset$, *where* $\varepsilon' = \dfrac{\varepsilon \min\{\varepsilon,\lambda\}}{2\sqrt{1+\varepsilon^2}}$ *and*
(3.1) $\quad \forall u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} \Longrightarrow |df|(u) > \varepsilon.$
Then there exists a continuous map $\eta : X \times [0,1] \longrightarrow X$ such that:

a) $d(\eta(u,t),u) \le \lambda t, \quad \forall t \in [0,1],\ \forall u \in X,$
b) $f(\eta(u,t)) \le f(u), \quad \forall t \in [0,1],\ \forall u \in X,$
c) if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \eta(u,t) = u, \quad \forall t \in [0,1]$
d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$,
e) $\forall\ t \in]0,1]$ *and* $\forall\ u \in f^c \cap C$ *we have* $f(\eta(u,t)) < c$.

Proof. First, we suppose that the function $f : X \longrightarrow \mathbb{R}$ is Lipschitz continuous with constant 1. We consider the set:
(3.2) $\quad C^* := \{u \in X | c - \varepsilon \le f(u) \le c + \varepsilon, d(u,C) \le \varepsilon\ \}.$
Obvious the set C^* is a closed subset of X, and is not empty from (3.0). We observe that $d(u,C^*) \le \varepsilon$ implies $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon) \cap C_{2\varepsilon}$.
In fact, since $|f(v) - f(u)| \le 1 \cdot d(u,v)$ for $\forall u, v \in X$ we obtain

$$-d(u,v) \le f(u) - f(v) \le d(u,v), \quad \forall v \in C^*.$$

Using this relation and the fact $d(u, C^*) \leq \varepsilon$ we get

$$c - 2\varepsilon \leq f(u) \leq c + 2\varepsilon.$$

In a similar way, it's easy to verify that $d(u, C^*) \leq \varepsilon$ implies $u \in C_{2\varepsilon}$ also.

Because $\varepsilon > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ from the above we obtain $|df|(u) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$, for all u s.t. $d(u, C^*) \leq \varepsilon$.
Now we can apply Theorem 2.4, for $C := C^*, \delta := \varepsilon, \sigma := \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$. We get a continuous function $\eta' : X \times [0, \varepsilon] \longrightarrow X$ which satisfies the conditions 1)-4) from Theorem 2.4. Let $\lambda_1 := \min\{\lambda, \varepsilon\}$ and define the function $\eta : X \times [0, 1] \longrightarrow X$ by $\eta(u, t) = \eta'(u, \lambda_1 t)$. The properties a) and b) are obvious. If $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and since f is a Lipschitz function with constant 1, we have $d(u, C^*) > \varepsilon$ and using 3) from Theorem 2.4 we get $\eta(u, t) = u$.
For the proof of d) we distinguish two cases:
(3.3) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) \geq c - \varepsilon'$ it follows that $u \in C^*$, hence we have

$$f(\eta(u,1)) = f(\eta'(u, \lambda_1)) \leq f(u) - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} \leq c + \varepsilon' - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} = c - \varepsilon'.$$

(3.4) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) < c - \varepsilon'$, then from b) we get

$$f(\eta(u,1)) \leq f(u) < c - \varepsilon'.$$

For the proof of e) we use also the 4) from Theorem 2.4.
Now we consider the general case. For this let $C^{**} = \{(u, \xi) \in epi(f) | \; u \in C\}$. The set $epi(f)$ is closed in $X \times \mathbb{R}$ and it follows that $epi(f)$ is a complete metric space with the metric d_{ep}. In the next we prove that for every $(u, \xi) \in epi(f)$ with $(u, \xi) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C^{**}_{2\varepsilon}$, we have $|d\mathcal{G}_f|(u, \xi) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
We distinguish two cases:
I) Let $f(u) = \xi$. In this case we have two subcases.
a) $|df|(u) < \infty$. If $(u, f(u)) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C^{**}_{2\varepsilon}$, then we get $u \in f^{-1}([c - 2\varepsilon, c+2\varepsilon])$ and $d_{ep}((u, f(u)), C^{**}) \leq 2\varepsilon$. Since $d(u, C) \leq d_{ep}((u, f(u)), C^{**}) \leq 2\varepsilon$ we get $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and using (3.1) it follows that $|df|(u) > \varepsilon$. Since $|df|(u) < \infty$ from Proposition 2.3 we have $|d\mathcal{G}_f|(u, f(u)) = \dfrac{|df|(u)}{\sqrt{1 + |df|^2(u)}}$ and using the fact that the function $x \mapsto \dfrac{x}{\sqrt{1+x^2}}$ is increasing we have $|d\mathcal{G}_f|(u, f(u)) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
b) If $|df|(u) = \infty$ using Proposition 2.3 we get $|d\mathcal{G}_f|(u, f(u)) = 1 > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
II) If $f(u) < \xi$, then from Proposition 2.3 we have $|d\mathcal{G}_f|(u, \xi) = 1 > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ also.
From these we get that if $(u, \xi) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C^{**}_{2\varepsilon}$ then $|d\mathcal{G}_f|(u, \xi) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

We apply the previous step for $X := epi(f)$, $f := \mathcal{G}_f$ and $C := C^{**}$, using the fact that $\mathcal{G}_f$ is Lipschitz continuous with constant 1. Of course $C^{**} \cap \mathcal{G}_f^{c+\varepsilon'} \cap (\mathcal{G}_f)_{c-\varepsilon'} \neq \emptyset$. Then there exists a continuous mapping $\overline{\eta} := (\overline{\eta_1}, \overline{\eta_2}) : epi(f) \times [0,1] \longrightarrow epi(f)$ such that the following hold:
(3.5) $d_{ep}((\overline{\eta}(u,\xi), t), (u,\xi)) \leq \lambda t, \quad \forall\ (u,\xi) \in epi(f), \forall\ t \in [0,1];$
(3.6) $\mathcal{G}_f(\overline{\eta}(u,\xi), t) = \overline{\eta}_2((u,\xi), t) \leq \xi = \mathcal{G}_f(u,\xi),$ for all$(u,\xi) \in epi(f)$, and $\forall\ t \in [0,1]$;
(3.7) $\overline{\eta}((u,\xi), t) = (u,\xi)$ for every $(u,\xi) \in epi(f), t \in [0,1]$ with$(u,\xi) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$;
(3.8) $\overline{\eta}(\mathcal{G}_f^{c+\varepsilon'} \cap C^{**}, 1) \subset \mathcal{G}_f^{c-\varepsilon'}$;
(3.9) $\mathcal{G}_f(\overline{\eta}((u,\xi), t)) < c$ for every $t \in]0,1]$ and $\forall\ (u,\xi) \in \mathcal{G}_f^c \cap C^{**}$.

We define the function $\eta : X \times [0,1] \longrightarrow X$ by
(3.10) $\eta(u,t) = \overline{\eta}_1((u, f(u)), t)$.
Because $\overline{\eta}$ takes its values in $epi(f)$, we have
(3.11) $f(\overline{\eta}_1((u,f(u)),t)) \leq \overline{\eta}_2((u,f(u)),t)$
From (3.5) we have:

$$d(\eta(u,t),u) = d(\overline{\eta}_1((u,f(u)),t),u) \leq$$

$$\leq [d^2((\overline{\eta}_1(u,f(u)),t),u) + (\overline{\eta}_2((u,f(u)),t) - f(u))^2]^{\frac{1}{2}} =$$
$$= d_{ep}(\overline{\eta}((u,f(u)),t),(u,f(u))) \leq \lambda t.$$

From the relations (3.6) and (3.11) we get

$$f(\eta(u,t)) = f(\overline{\eta}_1((u,f(u)),t)) \leq \overline{\eta}_2((u,f(u)),t) \leq f(u).$$

From $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$ we have $(u,f(u)) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}^{**}$. Therefore if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, then from (3.8) we get $\eta(u,t) = \overline{\eta}_1((u,f(u)),t) = u$.

If $f(u) \leq c + \varepsilon'$ then from (3.8) and (3.11) we get

$$f(\eta(u,1)) = f(\overline{\eta}_1(u,f(u)),1) \leq \overline{\eta}_2((u,f(u)),1) \leq c - \varepsilon'.$$

From (3.9) and (3.11) we get the relation e).

Definition 3.2 *Let B be a closed subset of graphF. We shall say that the class $\mathcal{F}$ of subsets of graphF is homotopy stable with boundary B, if the following conditions hold:*

a) every set of $\mathcal{F}$ contains B;

b) for every set $A \in \mathcal{F}$ and for every continuous $\eta : graphF \times [0,1] \longrightarrow graphF$ satisfying $\eta = id$ on $(graphF \times \{0\}) \cup (B \times [0,1])$ we have that $\eta(A,1) \in \mathcal{F}$.

Definition 3.3 *We say that the set $G \subset graphF$ is dual to $\mathcal{F}$ if G verifies the following conditions:*

1) $d_g(G,B) > 0$, where the d_g means the distance between B and G

2) $G \cap A \neq \emptyset$, *for all* $A \in \mathcal{F}$.
We define the following function $\pi_{\mathbb{R}} : graphF \longrightarrow \mathbb{R}$ given by $\pi_{\mathbb{R}}(u,c) = c$, where $(u,c) \in graphF$.
Theorem 3.4 *Let (X, d) be a complete metric space, $F : X \to \overline{\mathbb{R}}$ be a multi-valued mapping with closed graph and nonempty values. We consider a stable family $\mathcal{F}$ of subset of graphF with boundary B, and let G be a set dual to $\mathcal{F}$ which verifies the following condition*
(3.12) $\inf \pi_{\mathbb{R}}(G) \geq \inf_{A\in\mathcal{F}} \sup \pi_{\mathbb{R}}(A) = c$, *where the number c is finite.*
Let $\varepsilon \in (0, \frac{d_g(B,G)}{3})$. *Then for every* $A \in \mathcal{F}$ *which verifies the relation*
(3.13) $\sup \pi_{\mathbb{R}}(A \cap G_\varepsilon) \leq c + \dfrac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}$
there exists $(x_\varepsilon, c_\varepsilon) \in graphF$ *such that the following assertions holds:*
a) $c - 2\varepsilon \leq c_\varepsilon \leq c + 2\varepsilon$;
b) $|dF|(x_\varepsilon, c_\varepsilon) \leq \varepsilon$;
c) $d_g((x_\varepsilon, c_\varepsilon), A) \leq 2\varepsilon$;
d) $d_g((x_\varepsilon, c_\varepsilon), G) \leq 3\varepsilon$.
Proof. From the definition of the number c, it's result that there exists an $A \in \mathcal{F}$ such that we have:

$$\sup \pi_{\mathbb{R}}(A \cap G_\varepsilon) \leq c + \frac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}.$$

The properties a)-d) are equivalent with the following implication:

$$\exists (x_\varepsilon, c_\varepsilon) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{G_{3\varepsilon}} \cap \overline{A_{2\varepsilon}} \;\text{ such that }\; |dF|(x_\varepsilon, c_\varepsilon) \leq \varepsilon.$$

We suppose the contrary, i.e.

$$\forall (v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{G_{3\varepsilon}} \cap \overline{A_{2\varepsilon}} \;\text{ we have }\; |dF|(v,b) > \varepsilon.$$

We consider the set $C := \overline{A \cap G_\varepsilon}$. We have $C_{2\varepsilon} \subset \overline{A_{2\varepsilon}} \cap \overline{G_{3\varepsilon}}$ and
(3.14) $\forall\ (v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} :\ \ |dF|(v,b) > \varepsilon$.
If we use Proposition 2.6, then for every

$$(v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} :\ \ |d\mathcal{G}_F|(v,b) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}.$$

Since (X,d) is a complete metric space and $graphF$ is closed in $X \times \mathbb{R}$ we have that $(graphF, d_g)$ is a complete metric space. We need to prove that $C \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$, where $\varepsilon' = \dfrac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}$. We observe that $C \subset \mathcal{G}_F^{c+\varepsilon'}$ (from (3.13) and from the continuity of $\mathcal{G}_F$). Therefore it's remain to prove that $C \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$. If not, $\sup \pi_{\mathbb{R}}(\overline{A \cap G_\varepsilon}) \leq c - \varepsilon'$. From this, we get $c \leq \inf \pi_{\mathbb{R}}(G) \leq \inf \pi_{\mathbb{R}}(A \cap G) \leq \sup \pi_{\mathbb{R}}(A \cap G) \leq \sup \pi_{\mathbb{R}}(\overline{A \cap G_\varepsilon}) \leq c - \varepsilon'$, contradiction. Since the function $\mathcal{G}_F$ is continuous, we apply Theorem 3.1, and we get a continuous function $\eta := (\eta_1, \eta_2) : graphF \times [0,1] \to graphF$ which satisfies the following properties:

a') $d_g(\eta((v,b),t),(v,b)) \leq \varepsilon t$;
b') $\eta_2((v,b),t) \leq b$ for every $(v,b) \in graphF$, and $t \in [0,1]$;
c') if $(v,b) \notin \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$ then we have $\eta((v,b),t) = (v,b)$;
d') $\eta(\mathcal{G}_F^{c+\varepsilon'} \cap C, 1) \subset \mathcal{G}_F^{c-\varepsilon'}$.

Let $A_1 = \eta(A,1)$. From a') we have $\eta((v,b),0) = (v,b)$ for every $(v,b) \in graphF$. From the relation c') we have $\eta((v,b),t) = (v,b)$ for all $t \in [0,1]$ and for all $(v,b) \in B$, since we have $B \subset C_{graphF}C_{2\varepsilon}$. If this relation does not hold, then there exists $(v,b) \in B$ such that $(v,b) \in C_{2\varepsilon}$. But we have

$$2\varepsilon = 3\varepsilon - \varepsilon < d_g(B,G) - \varepsilon \leq d_g(B,G_\varepsilon) \leq d_g(B, A \cap G_\varepsilon) = d_g(B, \overline{A \cap G_\varepsilon}) \leq 2\varepsilon,$$

which is a contradiction. Therefore $B = \eta(B,1) \subset \eta(A,1) = A_1$. Using the definition of homotopy stable family, we have $A_1 = \eta(A,1) \in \mathcal{F}$. We have $\eta(A,1) \cap G \subset \eta(A \cap G_\varepsilon, 1)$. In fact, if $x \in \eta(A,1) \cap G$, then there exists $y \in A$ such that $x = \eta(y,1) \in G$. But from the relation a') we have that $d_g(\eta(y,1),y) \leq \varepsilon$, therefore $y \in G_\varepsilon$. Hence, $x = \eta(y,1) \in \eta(A \cap G_\varepsilon, 1)$. Using the relations d') and (3.13) we have

(3.15) $A_1 \cap G = \eta(A,1) \cap G \subset \eta(A \cap G_\varepsilon, 1) \subset \eta(\overline{A \cap G_\varepsilon} \cap \mathcal{G}_F^{c+\varepsilon'}, 1) \subset \mathcal{G}_F^{c-\varepsilon'}$.

From the relations (3.15) and (3.12) we get

$$c \leq \inf \pi_{\mathbb{R}}(G) \leq \inf \pi_{\mathbb{R}}(A_1 \cap G) \leq c - \varepsilon',$$

which is a contradiction.

An imediate consequence of this result is the following which will be used in the next section.

Corollary 3.5 *Let (X, d) be a complete metric space, $F : X \to \overline{\mathbb{R}}$ be a multivalued mapping with closed graph and nonempty values. We consider a homotopy stable family $\mathcal{F}$ with boundary B, and let G be a set dual to $\mathcal{F}$ which verifies the following condition*

$$(3.16) \qquad \inf \pi_{\mathbb{R}}(G) \geq \inf_{A \in \mathcal{F}} \sup \pi_{\mathbb{R}}(A) = c,$$

where the number c is finite. Then there exists a sequence $(u_n, c_n) \in graphF$, such that the following assertions holds:
a) $c_n \longrightarrow c$;
b) $|dF|(u_n, c_n) \longrightarrow 0$;
c) $d_g((u_n, c_n), G) \longrightarrow 0$.

Remark 3.6 Theorem 3.4 and Corollary 3.5 generalize a large class of location results and with the suitable $(PS)_c$ condition it's possible to state classical existence results of critical points, see the papers of H. Brézis and L. Nirenberg [1], G. Fang [9], N. Ghoussoub [12].

4 Changing the metric

In this section we extend for multivalued functional a result of Corvellec from the paper [6], which state a relation between the weak slopes of the multivalued functionals if we change the metric of the metric space. The main result of this section is the following.

Theorem 4.1 *Let (X, d) be a complete metric space, $F : X \longrightarrow \mathbb{R}$ a multivalued map with closed graph, $\tilde{A} \subset graphF$, nonempty subset and $\beta : [0, \infty[\longrightarrow]0, 1]$ be a continuous function. Then there exists a metric $\tilde{d}$ on $graphF$ which is topologically equivalent to d_g (denoted again by d) and the following assertions are true:*

a) For every $B \subset graphF$ we have $\tilde{d}(\tilde{A}, B) \geq \int_0^{d(B,\tilde{A})} \beta(t)dt$.

Therefore, if $\int_0^\infty \beta(t)dt = \infty$, then $(graphF, \tilde{d})$ is complete if and only if $(graphF, d)$ is complete.

b) If we denote by $|\tilde{d}F|$ the weak slope of F with respect to the metric $\tilde{d}$, then we have $|\tilde{d}F|(u,c) \geq \dfrac{|dF|(u,c)}{\beta(d((u,c),\tilde{A}))}$, for every $(u, c) \in graphF$.

Proof. We follow the proof from [6, Theorem 4.1]. The assertion a) is same as [6, Theorem 4.1] for $X := graphF$, $\beta : [0, \infty[\longrightarrow]0, 1]$, with

$$\tilde{d}((u, c), (v, b)) = \inf\{\int_0^1 \beta(d(\gamma(t), \tilde{A})) \cdot \|\gamma'(t)\|dt | \gamma \in \Gamma_{(u,c),(v,b)}\},$$

where d is the metric associated to the norm of Y; Y being the Banach space such that $graphF$ is isometrically embedded into, see [10, Theorem of Arens - Eells] and $\Gamma_{(u,c),(v,b)}$ is the set of C^1- paths $\gamma : [0, 1] \longrightarrow Y$ with $\gamma(0) = (u, c), \gamma(1) = (v, b)$.

b) Let $(u, c) \in graphF$ fixed, and assume that $|dF|(u, c) > \sigma > 0$. Let $\delta > 0$ and $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : B((u, c), \delta) \times [0, \delta] \longrightarrow graphF$ continuous such that

$$\|\mathcal{H}((v, b), t) - (v, b)\| \leq t \cdot \sqrt{1 + \sigma^2} \tag{4.1}$$

$$\mathcal{H}_2((v, b), t) \leq b - \sigma \cdot t. \tag{4.2}$$

Let $\varepsilon > 0$ be a positive number such that $\beta_\varepsilon(d((u, c), \tilde{A})) \leq 1$, where

$$\beta_\varepsilon(d((u,c),\tilde{A})) = \begin{cases} \beta(d((u,c),\tilde{A})) + \varepsilon, & \text{if } \beta(d((u,c),\tilde{A})) < 1 \\ 1, & \text{if } \beta(d((u,c),\tilde{A})) = 1. \end{cases}$$

We suppose that δ is sufficiently small, such that $\beta(d((v, b), \tilde{A})) \leq \beta_\varepsilon(d((u, c), \tilde{A}))$, for all $(v, b) \in B((u, c), \delta(1 + \sqrt{1 + \sigma^2}))$.

Let $0 < \tilde{\delta} \leq \delta\beta_\varepsilon(d((u, c), \tilde{A}))$ such that $\tilde{B}((u, c), \tilde{\delta}) \subset B((u, c), \delta)$, and let $\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2) : \tilde{B}((u, c), \tilde{\delta}) \times [0, \tilde{\delta}] \longrightarrow graphF$, defined by

$$\tilde{\mathcal{H}}((v, b), t) = \mathcal{H}((v, b), \frac{t}{\beta_\varepsilon(d((u, c), \tilde{A}))}).$$

For $((v, b), t) \in \tilde{B}((u, c), \tilde{\delta}) \times [0, \tilde{\delta}]$ and $s \in [0, 1]$ we define the curve γ by $\gamma(s) = (v, b) + s(\tilde{\mathcal{H}}((v, b), t) - (v, b))$. We observe that $\gamma(s) \in B((u, c), \delta(1+\sqrt{1 + \sigma^2}))$ for

every $s \in [0,1]$. In fact, since $(v,b) \in \tilde{B}((u,c),\tilde{\delta})$, follows that $\|(v,b)-(u,c)\| \leq \delta$ and
$\|\tilde{\mathcal{H}}((v,b),t)-(v,b)\| = \|\mathcal{H}((v,b),\frac{t}{\beta_\varepsilon(d((u,c),\tilde{A}))})-(v,b)\| \leq \frac{\tilde{\delta}}{\beta_\varepsilon(d((u,c),\tilde{A}))} \cdot \sqrt{1+\sigma^2} \leq \delta \cdot \sqrt{1+\sigma^2}$.
From the relations (4.1) and (4.2) we obtain:
$\tilde{d}(\tilde{\mathcal{H}}((v,b),t),(v,b)) \leq \int_0^1 \beta(d((\gamma(s),\tilde{A}))\cdot\|\gamma'(s)\|ds = \int_0^1 \beta(d((\gamma(s),\tilde{A}))\|\tilde{\mathcal{H}}((v,b),t)-(v,b)\| \leq \int_0^1 \beta(d(\gamma(s),\tilde{A})\cdot \frac{t}{\beta_\varepsilon(d((u,c),\tilde{A}))}\cdot\sqrt{1+\sigma^2}ds \leq t\cdot\sqrt{1+\sigma^2} \leq t\cdot \sqrt{1+(\frac{\sigma}{\beta_\varepsilon(d((u,c),\tilde{A}))})^2}$.
For the component $\tilde{\mathcal{H}}_2$ we have
$\tilde{\mathcal{H}}_2((v,b),t) = \mathcal{H}_2((v,b),\frac{t}{\beta_\varepsilon(d((u,c),\tilde{A}))}) \leq b - \frac{\sigma}{\beta_\varepsilon(d((u,c),\tilde{A}))}\cdot t$.

It follows that $|\tilde{d}F|(u,c) \geq \frac{\sigma}{\beta_\varepsilon(d((u,c),\tilde{A}))}$, hence $|\tilde{d}F|(u,c) \geq \frac{\sigma}{\beta(d((u,c),\tilde{A}))}$, since $\varepsilon>0$ is arbitrary small. From this we have $|\tilde{d}F|(u,c) \geq \frac{|dF|(u,c)}{\beta(d((u,c),\tilde{A}))}$.

5 Mountain Pass Theorem

In this section we give a generalization and a simplified proof of a result of Ghoussoub and Preiss for multivalued mappings which satisfies the Cerami (C) condition.
Theorem 5.1 *Let X be a Banach space, and $F : X \longrightarrow \overline{\mathbb{R}}$ a multivalued function with closed graph and nonempty values. Let $\mathcal{F}$ be a homotopy stable family of $graphF$ with boundary B and G a set dual to $\mathcal{F}$. We suppose that the condition (3.16) holds and the number c is finite. Then there exists a sequence $(u_n,c_n) \in graphF$, such that:*

a) $c_n \longrightarrow c$;

b) $(1+\|u_n\|)\cdot|dF|(u_n,c_n) \longrightarrow 0$;

c) $d((u_n,c_n),G) \longrightarrow 0$.

Proof. We apply Corollary 3.5 with the metric $\tilde{d}$, defined in Theorem 4.1, where $\beta(t) = \frac{1}{1+t}, t>0$. We observe that $\int_0^1 \beta(t)dt = \infty$, therefore $(graphF,\tilde{d})$ is a complete metric space, and from Theorem 4.1 b) we have that

$$|\tilde{d}F|(u,c) \geq \frac{|dF|(u,c)}{\beta(d((u,c),\tilde{A}))} = (1+d((u,c),\tilde{A}))\cdot|dF|(u,c),$$

where $\tilde{A}$ is a fixed subset of graph F.
From Corollary 3.5, follows the existence of a sequence $(u_n,c_n) \in graphF$ such that $c_n \longrightarrow c$, $|\tilde{d}F|(u_n,c_n) \to 0$ and $\tilde{d}((u_n,c_n),G) \to 0$.

We choose $\tilde{A} = (0, F(0))$. Using the inequality

$$d((u,c),\tilde{A}) = \sqrt{\|u\|^2 + dist(c, \pi_{\mathbb{R}}(\tilde{A}))} \geq \|u\|$$

and the fact that the metric $\tilde{d}$ is equivalent with the metric d, we get that $(1+\|u_n\|)\cdot|dF|(u_n,c_n) \longrightarrow 0$ and $d((u_n,c_n),G) \longrightarrow 0$.

Definition 5.2 *A closed subset $F_1 \subset graphF$ separates two point (u_0,c_0) and (u_1,c_1) in $graphF$, if (u_0,c_0) and (u_1,c_1) belong to disjoint connected components in $graphF \setminus F_1$.*

The main result of this section is the following result.

Theorem 5.3 *Let X be a Banach space, $F : X \longrightarrow \overline{\mathbb{R}}$ be a multivalued mapping with closed graph and nonempty values, and $(u_0,c_0),(u_1,c_1)$ two points in the same liniar connected component of $graphF$. Let*

$$\Gamma = \{ \;\; g \in \mathcal{C}([0,1], graphF) \mid g(0) = (u_0,c_0),\;\; g(1) = (u_1,c_1) \;\}$$

and we consider the number $\gamma := \inf\limits_{g\in\Gamma} \max\limits_{t\in[0,1]} \pi_{\mathbb{R}}(g(t))$ which is finite. Suppose that the set $D_\gamma = \{\; (x,b) \in graphF \;\|\;\; b \geq \gamma \;\}$ separates the points $(u_0,c_0),(u_1,c_1)$. Then there exists a sequence $(x_n,b_n) \in graphF$ such that the following assertions hold:

a) $b_n \longrightarrow \gamma$;
b) $(1+\|x_n\|)\cdot|dF|(x_n,b_n) \longrightarrow 0$;
c) $d((x_n,b_n),D_\gamma) \longrightarrow 0$.

Proof. We observe that $\mathcal{F} = \Gamma$ is homotopy stable with boundary $B = \{(u_0,c_0),(u_1,c_1)\}$. In fact, for every $g \in \Gamma$ we have from the definition $B \subset g([0,1])$ and for any $g \in \Gamma$ and $\eta \in \mathcal{C}(graphF \times [0,1], graphF)$, verifying $\eta = id$ on $(graphF \cdot \{0\}) \cup B \times [0,1]$ it follows that $g_1(t) = \eta(g(t),1)$ is continuous and $g_1(0) = \eta(g(0),1) = \eta((u_0,c_0),1) = (u_0,c_0)$ and $g_1(1) = (u_1,c_1)$. Therefore $g_1 \in \Gamma$. Moreover, the set D_γ is dual to $\mathcal{F}$. Indeed, since D_γ separates the points (u_0,c_0) and(u_1,c_1) we have that $D_\gamma \cap B = \emptyset$. Using the fact that the set $D_\gamma = \mathcal{G}_F^{-1}([\gamma,\infty))$ is closed and B is compact, we have $d(B,D_\gamma) > 0$. From the definition of γ we get that $D_\gamma \cap g([0,1]) \neq \emptyset$ for all $g \in \Gamma$. In conclusion we have that D_γ is dual to $\mathcal{F}$. Since $\inf \pi_{\mathbb{R}}(D_\gamma) \geq \gamma$, from Theorem 5.1 follows the existence of a sequence $(x_n,b_n) \in graphF$ such that

a) $b_n \longrightarrow \gamma$;
b) $(1+\|x_n\|)\cdot|dF|(x_n,b_n) \longrightarrow 0$;
c) $d((x_n,b_n),D_\gamma) \longrightarrow 0$.

The above result is an extension of Ghoussoub-Preiss result for multivalued mapping, see [8, Theorem 6., p.140].

Definition 5.4(Condition (C)) *Let X be a normed space and $F : X \longrightarrow \overline{\mathbb{R}}$ be a multivalued mapping with closed graph, $G \subset graphF$ a closed subset. We say that F satisfies the condition (C) at level $\gamma \in \mathbb{R}$ around G if for every sequence $(x_n,c_n) \in graphF$ which satisfy*

$$c_n \longrightarrow \gamma$$

$$d((x_n, c_n), G) \longrightarrow 0$$
$$(1 + \|x_n\|) \cdot |dF|(x_n, c_n) \longrightarrow o$$

has a convergent subsequence, which converge to a critical point of F on the level γ*, i.e.* $(x_n, c_n) \longrightarrow (x, c) \in G$*, and* $|dF|(x, \gamma) = 0$.

The following result is an immediate consequence of Theorem 5.3 and generalize the "Mountain Pass " theorem of Ambrosetti-Rabinowitz, see [8, Corollary 9., p. 145].

Theorem 5.5 *Let X be a Banach space,* $F : X \longrightarrow \overline{\mathbb{R}}$ *a multivalued mapping with closed graph and nonempty values. Let* $(u_0, c_0), (u_1, c_1)$ *two points in the same connected component of* $graphF$*, and define*

$$\Gamma = \{ \ \ g \in \mathcal{C}([0,1], graphF) \mid g(0) = (u_0, c_0), \ g(1) = (u_1, c_1) \ \}$$
$$\gamma := \inf_{g \in \Gamma} \max_{t \in [0,1]} \pi_{\mathbb{R}}(g(t)).$$

Assume that F satisfies the condition (C) at level γ *for graphF, and we have*

$$\gamma > \max\{c_0, c_1\}. \tag{5.1}$$

Then there exists a critical point of F at level γ*, i.e.there exists* $(x, \gamma) \in graphF$ *such that* $|dF|(x, \gamma) = 0$.

Proof. The relation (5.1) and the definition of γ implie, that $D_\gamma = \{ \ (x, b) \in graphF \ \mid \ b \geq \gamma \ \}$ separates the points (u_0, c_0) and (u_1, c_1). If we apply Theorem 5.3 with the condition (C) we get the assertion.

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