Bounded Imaginary Powers of Differential Operators on Manifolds with Conical Singularities

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Abstract. We study the minimal and maximal closed extension of a differential operator A on a manifold B with conical singularities, when A acts as an unbounded operator on weighted L_p-spaces over B, $1 < p < \infty$.

Under suitable ellipticity assumptions we can define a family of complex powers A^z, $z \in \mathbb{C}$. We also obtain sufficient information on the resolvent of A to show the boundedness of the pure imaginary powers.

Examples concern unique solvability and maximal regularity of the solution of the Cauchy problem $u' - \Delta u = f$, $u(0) = 0$, for the Laplacian on conical manifolds.

Contents

1. Introduction

Seeley's classical paper [**17**], published in 1967, showed in a striking way how pseudodifferential techniques could be applied to analyze complex powers of elliptic (pseudo-)differential operators on closed manifolds. Replacing the resolvent in the Dunford integral by a parameter-dependent parametrix, he obtained a representation of the powers that was precise enough to deduce a wealth of information on eigenvalue asymptotics, zeta functions, and index theory. Seeley also extended his results to differential boundary value problems. In 1971 he showed the boundedness of the purely imaginary powers on L_p-spaces, [**19**].

At that time the principal motivation for these studies was the description of interpolation spaces. An additional motivation for studying the behavior of imaginary powers came from Dore and

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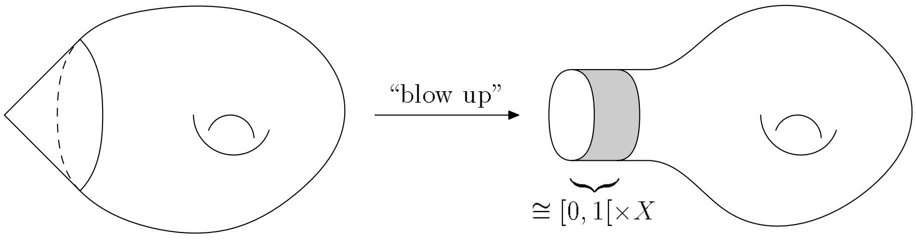
Venni's 1987 article [**3**], in which they showed how the boundedness of imaginary powers can be used to derive results on maximal regularity for evolution equations.

Meanwhile, bounded imaginary powers or even the existence of a bounded H^∞ calculus [**11**] have been established in many situations, e.g. for classes of differential operators on $\mathbb{R}^n$ and smooth manifolds [**2**], boundary value problems on bounded and certain unbounded domains in $\mathbb{R}^n$, [**4**], [**12**], [**22**], as well as for operators in Boutet de Monvel's calculus [**21**].

We shall focus here on the case of a manifold with a conical singularity. This is a Hausdorff space, B, that is a smooth manifold outside the singular point, while, close to this point it has the structure of a cone with smooth, closed cross-section X. Blowing up B near its tip, we obtain a manifold $\mathbb{B}$ with boundary $\partial\mathbb{B} = X$:



Near the boundary, we fix a splitting of coordinates $(t, x) \in [0, 1[\times X$. Rather than on B, the analysis will be performed on $\mathbb{B}$ (respectively the interior of $\mathbb{B}$). We consider so-called *cone* or *Fuchs-type* differential operators, i.e., operators which close to the boundary are of the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \tag{1.1}$$

where each $a_j \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$ is a smooth family of differential operators on the cross-section. Such an A acts as an unbounded operator $A : \mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}) \subset \mathcal{H}^{0,\gamma}_p(\mathbb{B}) \to \mathcal{H}^{0,\gamma}_p(\mathbb{B})$, where the space $\mathcal{H}^{0,\gamma}_p(\mathbb{B})$ away from the boundary coincides with $L_p(\mathbb{B})$ and near the boundary with

$$t^{\gamma-\frac{n+1}{2}} L_p([0, 1[\times X, \tfrac{dt}{t} dx), \qquad n = \dim X.$$

Here, $1 < p < \infty$, and γ is an arbitrary real number. The particular choice of $\gamma = \gamma_p = (n+1)(\frac{1}{2}-\frac{1}{p})$ leads to the natural space of p-integrable functions on B, i.e., $L_p(B) = \mathcal{H}^{0,\gamma_p}_p(\mathbb{B})$. This is justified by the fact that a change to polar coordinates shows the equivalence $L_p(\mathbb{R}^{n+1}) = t^{-\frac{n+1}{p}} L_p(\mathbb{R}_+ \times S^n, \frac{dt}{t} d\varphi)$.

Let $\Lambda_\Delta = \Lambda_\Delta(\theta)$ denote a closed sector in the complex plane, symmetric about the negative real half-axis and of aperture $2(\pi - \theta)$ for some $0 < \theta < \pi$. We find conditions (Definition 4.5) on A, which depend on $\gamma \in \mathbb{R}$ but not on $1 < p < \infty$, that ensure the following:

i) the *closure* $A_{\min}$ of A has no spectrum in $\Lambda_\Delta \cap \{|\lambda| > R\}$;
ii) the resolvent satisfies the uniform estimate $\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} \le c_p\, |\lambda|^{-1}$.

Moreover, we obtain very precise information on the structure of the resolvent. For this and i) see Theorem 4.6; ii) is shown in Proposition 4.7. We also give conditions (Remark 6.4) implying that the *maximal extension* $A_{\max}$ satisfies statements analogous to i) and ii). Note that the choice of the symmetry axis $]-\infty, 0]$, which in classical terms [**1**], [**19**], is called a ray of minimal growth for A, is inessential for our purpose and only done for convenience. The case of an arbitrary symmetry axis $\{te^{i\varphi} \mid t \ge 0\}$ can be reduced to our situation, replacing A by $e^{-i\varphi}A$.

Since $A_{\min}$ (respectively $A_{\max}$) in the above case has compact resolvent, any "keyhole" region Λ, consisting of the sector Λ_Δ and an arbitrarily small ball around zero, only contains finitely many elements of the spectrum. Assuming that zero is the only spectral point in the keyhole (or, alternatively, shrinking the angle of the sector and possibly rotating A a little), we define *complex powers* $A^z_{\min}$ (respectively $A^z_{\max}$) for all complex z with negative real part. This is done in terms of a Dunford integral, integrating the resolvent against λ^z along the boundary of the keyhole. Using the specific structure of the resolvent, we show (Theorem 5.1) that

iii) $\|A^z_{\min}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} \le c_p\, e^{|\operatorname{Im} z|\theta}$ uniformly for all z with $|\operatorname{Re} z|$ sufficiently small

as well as the analogous estimate for $A^z_{\max}$ (Theorem 6.3). Consequently, the purely imaginary powers $A^{iy}_{\min}$ (respectively $A^{iy}_{\max}$), $y \in \mathbb{R}$, exist as suitable limits and satisfy an estimate as in iii).

It should be noted that both the construction of the complex powers and the boundedness of the imaginary powers only rely on the information about the resolvent provided by Theorem 4.6. Our conclusions therefore carry over to all situations where the resolvent has this structure.

The key to the above described results is, similar to Seeley's classical concept, to view $\lambda - A$ as an element of a calculus of parameter-dependent pseudodifferential operators on $\mathbb{B}$, and to express $(\lambda - A)^{-1}$ within this calculus. In our context, the appropriate calculus is Schulze's parameter-dependent cone algebra, cf. for example [**15**], [**5**]. The conditions we impose on A are, more or less, ellipticity conditions on $\lambda - A$ within this calculus. We require three associated objects not to have spectrum in the sector Λ_Δ. The first is the usual homogeneous principal symbol of A, defined on the cotangent bundle over the interior of $\mathbb{B}$. The second is the so-called *rescaled symbol*, which reflects the behavior of the principal symbol near the boundary. The third is the so-called *model cone* operator $\widehat{A}$, which acts as an unbounded operator in Sobolev spaces on the infinite cylinder $\mathbb{R}_+ \times X$. It is induced by freezing the coefficients of A at the boundary, i.e., using the notation from (1.1), $\widehat{A} = t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j$.

In order to separate the more general functional-analytic issues from the specific difficulties related to conical singularities, we give a review of several basic facts about complex powers of unbounded operators on a Banach space in Section 2, while in Section 3 we briefly discuss Fuchs-type operators. Sections 4, 5 and 6 are devoted to the proof of the results stated above.

In Section 7 we treat an example and show how our work can be combined with that of Dore and Venni to obtain results on existence and regularity for the non-homogeneous Cauchy problem in $L_p(B)$:

$$u'(t) - \Delta u(t) = f(t), \qquad u(0) = 0.$$

Here, Δ is the Laplace-Beltrami operator for a Riemannian metric with a conical degeneracy.

The paper ends with an appendix relating the structure of the resolvent as we use it to that given in earlier work by Schulze and that of Gil [**6**]. Moreover, we collect a few definitions and notions in Section 9.

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2. Complex powers of operators in a Banach space

Let us recall some well-known facts on complex powers of a closed, densely defined operator

$$A : \mathcal{D}(A) \subset F \longrightarrow F$$

in a Banach space F, cf. for example [**19**]. We denote by $\Lambda = \Lambda(\delta, \theta)$ the keyhole region

$$\Lambda(\delta, \theta) = \{\lambda \in \mathbb{C} \mid |\lambda| \le \delta \text{ or } |\arg \lambda| \ge \theta\}$$

with $\delta > 0$ and $0 < \theta < \pi$. We assume that

(A1) The spectrum of A has empty intersection with $\Lambda \setminus \{0\}$.
(A2) $\|(\lambda - A)^{-1}\|_{\mathcal{L}(F)} |\lambda|$ is uniformly bounded for large $\lambda \in \Lambda$.

If $\mathcal{C}$ is the parametrization of the boundary of Λ, cf. (9.1), we let

$$A^z = \frac{1}{2\pi i}\int_{\mathcal{C}} \lambda^z(\lambda - A)^{-1}\,d\lambda, \qquad z \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}. \tag{2.1}$$

Here, λ^z is defined via the logarithm $\log\lambda = \log|\lambda| + i\arg\lambda$, where $-\pi < \arg\lambda < \pi$. Since the integrand is $O(|\lambda|^{-1+\operatorname{Re} z})$, the integral is absolutely convergent, and thus (2.1) defines continuous operators $A^z \in \mathcal{L}(F)$. The notation A^z should be viewed with a little care, since A^{-1} in general is not the inverse of A, which is not required to exist.

Remark 2.1. *Under conditions* (A1), (A2)*, the function* $z \mapsto A^z : \mathbb{H} \to \mathcal{L}(F)$ *is holomorphic and satisfies the semi-group property*

$$A^z A^w = A^{z+w}, \qquad z, w \in \mathbb{H}.$$

If one furthermore imposes that for some positive constant c

(A3) $\|A^z\|_{\mathcal{L}(F)}$ is uniformly bounded in the rectangle $-c \le \operatorname{Re} z < 0$, $|\operatorname{Im} z| \le k$ for any $k \in \mathbb{N}$,

then the limits of A^z for $z \to iy$ exist for any real y. More precisely:

Remark 2.2. *Under conditions* (A1), (A2)*, and* (A3)*, the limits*

$$A^{iy}f = \lim_{\mathbb{H}\ni z \to iy} A^z f$$

exist for any real number y and any $f \in F$, and thus define operators $A^{iy} \in \mathcal{L}(F)$. Furthermore $A^{iy}f = A^{-1+iy}Af$ for $f \in \mathcal{D}(A)$. In particular, if we set

$$E_0 = \frac{1}{2\pi i}\int_{|\lambda|=\delta} (\lambda - A)^{-1}\,d\lambda,$$

then E_0 is a projection in F and $A^0 = 1 - E_0$.

Remark 2.2 could be rephrased as follows: Under conditions (A1), (A2), and (A3), the operators

$$T^z := A^z + E_0, \qquad z \in \mathbb{H},$$

form an analytic semi-group (with $\lim_{z\to 0} T^z f = f$ for any $f \in F$) and there exist constants $c \ge 1$ and $\omega \ge 0$ such that

$$\|T^z\|_{\mathcal{L}(F)} \le c\,e^{\omega|z|}, \qquad z \in \mathbb{H}.$$

Moreover, $(1 - E_0)A + E_0 : \mathcal{D}(A) \to F$ is an isomorphism, whose inverse is T^{-1}.

In concrete situations the problem is to analyze whether an operator A satisfies conditions (A1), (A2), (A3), and then to find the best possible constant ω. Fundamental works on this topic are due to Seeley [**17**], [**18**], [**19**], where he gives criteria ensuring that a differential operator on a compact manifold (with boundary) has these properties. The main object of the present paper is to give such criteria for differential operators on manifolds with conical singularities.

3. Cone differential operators

The natural class of differential operators on $\mathbb{B}$ is the algebra of *cone differential* or *Fuchs-type* operators, cf. [**5**], [**9**], i.e. those A which are near the boundary $\partial\mathbb{B} = X$ of the form

$$A = t^{-\mu}\sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j; \tag{3.1}$$

here $a_j \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+, \operatorname{Diff}^{\mu-j}(X))$ are functions smooth up to the boundary with values in the differential operators on X. We can rewrite (3.1) as

$$A = t^{-\mu}\operatorname{op}_M^{\gamma+\mu-\frac{n}{2}}(f), \qquad f(t,z) = \sum_{j=0}^{\mu} a_j(t)z^j, \tag{3.2}$$

where the Mellin pseudodifferential operator is defined by

$$[\mathrm{op}_M^{\gamma+\mu-\frac{n}{2}}(f)u](t) = \int_{\mathrm{Re}\, z=\frac{n+1}{2}-\gamma-\mu} t^{-z} f(t,z)(\mathcal{M}u)(z)\, dz, \qquad u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}). \tag{3.3}$$

Here, $\gamma \in \mathbb{R}$ is arbitrary, and (3.3) is independent of the choice of γ. We keep γ in the notation, since we shall consider extensions of A to different weighted Sobolev spaces, the weight being given by γ:

Definition 3.1. *For $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, and $1 < p < \infty$ we introduce $\mathcal{H}^{s,\gamma}_p(\mathbb{B})$ as the space of all functions $u \in H^s_{p,\mathrm{loc}}(\mathrm{int}\,\mathbb{B})$ such that*

$$t^{\frac{n+1}{2}-\gamma}(t\partial_t)^k \partial_x^\alpha(\omega u)(t,x) \;\in\; L_p(\mathbb{R}_+ \times X, \tfrac{dt}{t}dx) \qquad \forall\, k + |\alpha| \le s$$

for some cut-off function $\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$.

The definition of the Banach spaces $\mathcal{H}^{s,\gamma}_p(\mathbb{B})$ naturally extends to real s. Then A induces, for any s, γ, and p, continuous operators

$$A : \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B}) \longrightarrow \mathcal{H}^{s,\gamma}_p(\mathbb{B}).$$

With A we associate three symbols. The first is the usual *homogeneous principal symbol* $\sigma^\mu_\psi(A) \in \mathcal{C}^\infty(T^*(\mathrm{int}\,\mathbb{B}) \setminus 0)$, which, in coordinates near the boundary, is

$$\sigma^\mu_\psi(A)(t,x,\tau,\xi) = t^{-\mu}\sum_{j=0}^{\mu} \sigma^{\mu-j}_\psi(a_j)(t,x,\xi)(it\tau)^j \tag{3.4}$$

Dropping the factor $t^{-\mu}$, replacing $t\tau$ by τ, and inserting $t = 0$, we obtain

$$\tilde\sigma^\mu_\psi(A)(x,\tau,\xi) := \sum_{j=0}^{\mu} \sigma^{\mu-j}_\psi(a_j)(0,x,\xi)(i\tau)^j,$$

which yields the *rescaled symbol* of A,

$$\tilde\sigma^\mu_\psi(A) \;\in\; \mathcal{C}^\infty((T^*X \times \mathbb{R}) \setminus 0). \tag{3.5}$$

The third one is the *conormal symbol*

$$\sigma^\mu_M(A)(z) = f(0,z) = \sum_{j=0}^{\mu} a_j(0) z^j, \tag{3.6}$$

a function of $z \in \mathbb{C}$ with values in the differential operators on X.

Remark 3.2. *The operator A is called elliptic with respect to the weight $\gamma+\mu$, if both the homogeneous principal symbol and the rescaled symbol never vanish, and*

$$\sigma^\mu_M(A)(z) : H^s(X) \to H^{s-\mu}(X), \qquad \mathrm{Re}\, z = \tfrac{n+1}{2} - \gamma - \mu, \tag{3.7}$$

is an isomorphism for all z on that line.

It can be shown, [**14**] *Theorem 3.13, that A is elliptic if and only if the operators $A : \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B}) \longrightarrow \mathcal{H}^{s,\gamma}_p(\mathbb{B})$ are Fredholm for any s and p.*

We shall consider A as the operator

$$A : \mathcal{D}(A) = \mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B}) \subset \mathcal{H}^{0,\gamma}_p(\mathbb{B}) \longrightarrow \mathcal{H}^{0,\gamma}_p(\mathbb{B}). \tag{3.8}$$

Remark 3.3.
a) *In case A is elliptic with respect to $\gamma+\mu$, (3.8) is the closure of A considered on the domain $\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B})$.*
b) *By the spectral invariance of the cone algebra,* [**14**] *Theorem 3.14, the spectrum of A is independent of $1 < p < \infty$.*

With A we associate the model cone operator, which acts in Sobolev spaces on the infinite cylinder

$$X^\wedge := \mathbb{R}_+ \times X. \tag{3.9}$$

Let (t,x) denote cylindrical coordinates on $X^\wedge$. Then $H^s_{p,\mathrm{cone}}(X^\wedge)$ is the space of all distributions u whose push-forward under conical coordinates (t,tx) belongs to $H^s_p(\mathbb{R}^{1+n})$ (for details see [**13**], Section 4.2).

Definition 3.4. *For $s,\gamma \in \mathbb{R}$ and $1 < p < \infty$ the spaces $\mathcal{K}^{s,\gamma}_p(X^\wedge)$ consist of all distributions $u \in H^s_{p,\mathrm{loc}}(X^\wedge)$ satisfying, for some cut-off function $\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$,*

$$\omega u \in \mathcal{H}^{s,\gamma}_p(\mathbb{B}), \qquad (1-\omega)u \in H^s_{p,\mathrm{cone}}(X^\wedge).$$

Freezing the coefficients of A at $t=0$, we obtain the model cone operator $\widehat{A}$,

$$\widehat{A} = t^{-\mu}\sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j : \mathcal{K}^{\mu,\gamma+\mu}_p(X^\wedge) \longrightarrow \mathcal{K}^{0,\gamma}_p(X^\wedge). \tag{3.10}$$

Remark 3.5. a) *If A is elliptic with respect to the weight $\gamma+\mu$ and satisfies the ellipticity condition* (E1) *introduced below, then it can be shown that* (3.10) *is the closure of $\widehat{A}$ considered on the domain $\mathcal{C}^\infty_{\mathrm{comp}}(X^\wedge)$.*

b) *If we set $\widehat{a}(\lambda) = \lambda - \widehat{A}$ with $\widehat{A}$ from* (3.10)*, then $\widehat{a}(\lambda)$ corresponds to the so-called* principal edge symbol *of $\lambda - A$, if we view $\lambda - A$ as a constant coefficient edge symbol in the framework of Schulze's theory of pseudodifferential operators on manifolds with edges, cf. for example* [**5**].

It is worth mentioning that $\widehat{a}(\lambda) = \lambda - \widehat{A}$ is a homogeneous function in a specific way. Namely if we define for $\varrho > 0$

$$\kappa_\varrho : \mathcal{C}^\infty_{\mathrm{comp}}(X^\wedge) \to \mathcal{C}^\infty_{\mathrm{comp}}(X^\wedge), \quad (\kappa_\varrho u)(t,x) = \varrho^{\frac{n+1}{2}} u(\varrho t, x), \tag{3.11}$$

then these operators extend by continuity to isomorphisms in $\mathcal{L}(\mathcal{K}^{s,\gamma}_p(X^\wedge))$ and

$$\widehat{a}(\varrho^\mu \lambda) = \varrho^\mu \, \kappa_\varrho \, \widehat{a}(\lambda) \, \kappa_\varrho^{-1}. \tag{3.12}$$

In particular, $\mathrm{spec}(\widehat{A})$ is a closed conic subset of the complex plane.

Near the boundary of $\mathbb{B}$ we can write $\lambda - A = t^{-\mu}\mathrm{op}_M^{\gamma+\mu-\frac{n}{2}}(h)(\lambda)$ with the parameter-dependent Mellin symbol

$$h(t,z,\lambda) = \tilde{h}(t,z,t^\mu\lambda), \qquad \tilde{h}(t,z,\lambda) = \lambda - f(t,z), \tag{3.13}$$

and f from (3.2).

4. The resolvent of cone differential operators

To describe the structure of the resolvent we recall some elements from the theory of parameter-dependent cone pseudodifferential operators, starting with the smoothing remainders of the calculus. To this end we introduce a family of Fréchet spaces of smooth functions on $\mathrm{int}\,\mathbb{B}$ and $X^\wedge$, respectively.

Definition 4.1. *For $\gamma \in \mathbb{R}$ we let $\mathcal{C}^{\infty,\gamma}(\mathbb{B})$ denote the space of all $u \in \mathcal{C}^\infty(\mathrm{int}\,\mathbb{B})$ such that*

$$\sup_{0<t<1} t^{\frac{n+1}{2}-\gamma} |||\log^l t (t\partial_t)^k(u)(t,\cdot)||| < \infty \qquad \forall\, k,l \in \mathbb{N}_0 \tag{4.1}$$

for any semi-norm $|||\cdot|||$ of $\mathcal{C}^\infty(X)$. Similarly, $\mathcal{S}^\gamma_0(X^\wedge)$ is the space of all $u \in \mathcal{C}^\infty(X^\wedge)$ which are rapidly decreasing as $t \to \infty$ and satisfy (4.1).

We shall say that an operator G has a kernel k with respect to the $\mathcal{H}_2^{0,0}(\mathbb{B})$-scalar product if

$$(Gu)(y) = \langle k(y,\cdot), \overline{u}\rangle_{\mathcal{H}_2^{0,0}(\mathbb{B})} = \int_{\mathbb{B}} k(y,y')u(y')\,t(y')^n dy', \qquad u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}),$$

where t denotes a boundary defining function on $\mathbb{B}$ and dy' refers to a density on $2\mathbb{B}$, the double of $\mathbb{B}$. We shall use the analogous notion for operators on $X^\wedge$, based on the scalar product of $\mathcal{K}_2^{0,0}(X^\wedge) = L_2(X^\wedge, t^n dtdx)$.

Definition 4.2. *An operator-family $G = G(\lambda)$, $\lambda \in \Lambda$, belongs to $C_G^{-\infty}(\mathbb{B};\Lambda,\gamma)$, $\gamma \in \mathbb{R}$, if there exists an $\varepsilon = \varepsilon(G) > 0$ such that $G(\lambda)$ has a kernel $k(\lambda) = k(\lambda,\cdot,\cdot)$ with respect to the $\mathcal{H}_2^{0,0}(\mathbb{B})$-scalar product and*

$$k(\lambda,y,y') \in \mathcal{S}(\Lambda, \mathcal{C}^{\infty,\gamma+\varepsilon}(\mathbb{B}_y)\,\widehat{\otimes}_\pi\,\mathcal{C}^{\infty,-\gamma+\varepsilon}(\mathbb{B}_{y'})),$$

cf. (9.3); $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. $C_G^{-\infty}(\mathbb{B};\Lambda,\gamma)$ is the residual class of the calculus. For every choice of s, p, and λ, the operator $G(\lambda)$ maps $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ into $\mathcal{C}^{\infty,\gamma}(\mathbb{B})$. For the description of the resolvent we shall need another class of operator-families. For each fixed λ, they are smoothing over $X^\wedge$, yet they have a finite order in λ:

Definition 4.3. *Let $\gamma,\mu \in \mathbb{R}$ and $d > 0$. We define $R_G^{\mu,d}(X^\wedge;\Lambda,\gamma)$ as the space of all operator-families $G = G(\lambda)$ that have a kernel with respect to the $\mathcal{K}_2^{0,0}(X^\wedge)$-scalar product of the form*

$$k(\lambda,t,x,t',x') = [\lambda]^{\frac{n+1}{d}}\tilde{k}(\lambda,[\lambda]^{\frac{1}{d}}t,x,[\lambda]^{\frac{1}{d}}t',x'),$$

where $[\cdot]$ is a smoothed norm-function (i.e., $[\cdot]$ is smooth, positive on $\mathbb{C}$ and $[\lambda] = |\lambda|$ for large λ) and for some $\varepsilon = \varepsilon(G) > 0$

$$\tilde{k}(\lambda,t,x,t',x') \in S_{cl}^{\frac{\mu}{d}}(\Lambda)\,\widehat{\otimes}_\pi\,\mathcal{S}_0^{\gamma+\varepsilon}(X^\wedge_{(t,x)})\,\widehat{\otimes}_\pi\,\mathcal{S}_0^{-\gamma+\varepsilon}(X^\wedge_{(t',x')}).$$

In this case, $G(\lambda)$ maps $\mathcal{K}_p^{s,\gamma}(X^\wedge)$ into $\mathcal{S}_0^\gamma(X^\wedge)$ for any s and p. See also the Appendix for more information on such operator-families. Trivially, a symbol $a \in S_{cl}^{\frac{\mu}{d}}(\Lambda)$ satisfies the estimate

$$|a(\lambda)| \le c\,(1+|\lambda|)^{\frac{\mu}{d}}, \qquad \lambda \in \Lambda. \tag{4.2}$$

Recall from Section 3 that if A is a cone differential operator, then $\lambda - A$ can be written in terms of Mellin symbols taking values in the differential operators on X, cf. (3.13). In that case the Mellin symbol is a polynomial in z. A general Mellin symbol is an entire function with values in the pseudodifferential operators on X; more precisely:

Definition 4.4. *For $\mu \in \mathbb{R}$ and $d > 0$ let $M_{\mathcal{O}}^{\mu,d}(X;\Lambda)$ denote the space of all functions $\tilde{g}(z,\lambda)$, which are holomorphic in $z \in \mathbb{C}$ with values in $L^{\mu,d}(X;\Lambda)$, and for which*

$$\tilde{g}_\beta(\tau,\lambda) := \tilde{g}(\beta+i\tau,\lambda) \ \in\ L^{\mu,d}(X;\mathbb{R}_\tau\times\Lambda)$$

is locally bounded as a function of $\beta \in \mathbb{R}$. This is a Fréchet space in a canonical way.

Let us now state the ellipticity assumptions on A, which ensure the existence of its resolvent in a keyhole region:

Definition 4.5. *We call A elliptic with respect to the weight $\gamma+\mu$ and the sector Λ_Δ, cf. (9.2), if the following two conditions are satisfied:*

- (E1) *Both the homogeneous principal symbol $\sigma_\psi^\mu(A)$ and the rescaled symbol $\tilde{\sigma}_\psi^\mu(A)$, cf. (3.5), do not take values in Λ_Δ,*
- (E2) *the model cone operator $\widehat{A}$, acting as in (3.10), has no spectrum in $\Lambda_\Delta \setminus \{0\}$.*

If conditions (E1) and (E2) are satisfied, they automatically hold for a slightly larger keyhole region (by closedness of the spectrum, compactness of $\mathbb{B}$, and the homogeneity of the rescaled symbol, the homogeneous principal symbol, as well as $\widehat{a}(\lambda)$, cf. (3.12)). Moreover, one can show that condition (E2) implies that (3.7) is a family of isomorphisms.

We would like to point out that, although the above conditions seem to be quite strong, it follows from more general considerations that they are essentially necessary.

Under conditions (E1) and (E2) we can now describe the resolvent of A:

THEOREM 4.6. *If A is elliptic with respect to $\Lambda_\Delta = \Lambda_\Delta(\theta)$ and $\gamma+\mu$, then A has no spectrum in $\Lambda_\Delta \cap \{|\lambda| > R\}$ for some $R > 0$, and for large $\lambda \in \Lambda_\Delta$*

$$(\lambda - A)^{-1} = \sigma\left\{t^\mu \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)(\lambda) + G(\lambda)\right\}\sigma_0 + (1-\sigma)P(\lambda)(1-\sigma_1) + G_\infty(\lambda), \tag{4.3}$$

where $\sigma, \sigma_0, \sigma_1 \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ are cut-off functions satisfying $\sigma_1\sigma = \sigma_1$, $\sigma\sigma_0 = \sigma$, and there exists a $\delta > 0$ such that, for $\Lambda = \Lambda(\delta,\theta)$,

i) *$g(t,z,\lambda) = \tilde{g}(t,z,t^\mu\lambda)$ with $\tilde{g} \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{\mu,d}(X;\Lambda))$,*
ii) *$P(\lambda) \in L^{-\mu,\mu}(\mathrm{int}\,\mathbb{B};\Lambda)$, cf. (9.4),*
iii) *$G(\lambda) \in R_G^{-\mu,\mu}(X^\wedge;\Lambda,\gamma)$, and $G_\infty(\lambda) \in C_G^{-\infty}(\mathbb{B};\Lambda,\gamma)$.*

In view of the fact that A has compact resolvent (recall that the embeddings $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{r,\varrho}(\mathbb{B})$ are compact provided $s > r$ and $\gamma > \varrho$), only finitely many points of the spectrum of A will lie in Λ. Thus, after possibly rotating A a little and shrinking the keyhole Λ, we can assume that A has no spectrum in Λ, except perhaps 0.

Theorem 4.6 follows from the parametrix construction in the parameter-dependent cone algebra given in [**6**], Theorems 3.2, 3.4, cf. also [**5**], Section 9.3.3, Theorem 6. An important observation we can draw from this theorem is a norm estimate of the resolvent:

PROPOSITION 4.7. *Under the assumptions of Theorem 4.6 there exists a constant $c_p \geq 0$ such that for all sufficiently large $\lambda \in \Lambda$*

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p\,|\lambda|^{-1}.$$

PROOF. We first reduce the case of arbitrary γ to the special case $\gamma = \gamma_p = (n+1)(\frac{1}{2} - \frac{1}{p})$. To this end let $b \in \mathcal{C}^\infty(\mathrm{int}\,\mathbb{B})$ be a positive function such that $b(t,x) = t^\nu$, $\nu = \gamma_p - \gamma$, for all $(t,x) \in [0,1] \times X$. Multiplication by b induces isomorphisms $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s,\gamma_p}(\mathbb{B})$ with inverse induced by b^{-1}. Therefore,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \sim \|b(\lambda - A)^{-1}b^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma_p}(\mathbb{B}))}.$$

But for large $|\lambda|$,

$$b(\lambda - A)^{-1}b^{-1} = \sigma\left\{t^\mu \mathrm{op}_M^{\gamma_p-\frac{n}{2}}(T^{-\nu}g)(\lambda) + t^\nu G(\lambda)t^{-\nu}\right\}\sigma_0 + (1-\sigma)bP(\lambda)b^{-1}(1-\sigma_1) + bG_\infty(\lambda)b^{-1},$$

where $T^{-\nu}g(t,z,\lambda) = g(t,z-\nu,\lambda)$. Since $T^{-\nu}g$ and $bP(\lambda)b^{-1}$ are of the same quality as g and $P(\lambda)$, respectively, and $t^\nu G(\lambda)t^{-\nu} \in R_G^{-\mu,\mu}(X^\wedge;\Lambda,\gamma_p)$ and $bG_\infty(\lambda)b^{-1} \in C_G^{-\infty}(\mathbb{B};\Lambda,\gamma_p)$, we can assume from the very beginning that $\gamma = \gamma_p$.

The term $G_\infty(\lambda)$ certainly behaves in the right way, since it is rapidly decreasing in λ. Also the term $(1-\sigma)P(\lambda)(1-\sigma_1)$ is good by the standard Calderon-Vaillancourt theorem. The two remaining terms $t^\mu \mathrm{op}_M^\gamma(g)(\lambda)$ and $G(\lambda)$ we shall consider in the spaces

$$\mathcal{K}_p^{0,\gamma_p}(X^\wedge) = L_p(\mathbb{R}_+ \times X, t^n dtdx).$$

If κ_ϱ is the group action from (3.11), then $\|\kappa_\varrho\|_{\mathcal{L}(\mathcal{K}_p^{0,\gamma_p}(X^\wedge))} = \varrho^{\gamma_p}$ for all $\varrho > 0$. Hence for an arbitrary operator $T \in \mathcal{L}(\mathcal{K}_p^{0,\gamma_p}(X^\wedge))$ we have

$$\|T\|_{\mathcal{L}(\mathcal{K}_p^{0,\gamma_p}(X^\wedge))} = \|\kappa_\varrho^{-1}T\kappa_\varrho\|_{\mathcal{L}(\mathcal{K}_p^{0,\gamma_p}(X^\wedge))}.$$

Now let $G(\lambda)$ have a kernel $k(\lambda)$ as described in Definition 4.3 (with μ replaced by $-\mu$ and $d=\mu$). Then the operator-norm of $G(\lambda)$ is the same as that of $\kappa^{-1}_{[\lambda]^{1/\mu}}G(\lambda)\kappa_{[\lambda]^{1/\mu}}$, which has the kernel $\tilde{k}(\lambda,t,x,t',x')$. But this kernel is $O(|\lambda|^{-1})$ in λ, cf. (4.2). To treat the last term we can pass to local coordinates, i.e. we assume $X=\mathbb{R}^n$ and

$$\tilde{g}_{\frac{1}{2}-\gamma_p}(t,x,\tau,\xi,\lambda)\;\in\;S^{-\mu,\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^n\times\mathbb{R}^{1+n};\Lambda).$$

By a tensor product argument we can assume that $\tilde{g}$ is independent of (t,x). Conjugating with $\kappa_{[\lambda]^{1/\mu}}$, we have to show that

$$\mathrm{op}_M^{\gamma_p-\frac{n}{2}}(g')(\lambda),\qquad g'_{\frac{1}{2}-\gamma_p}(t,\tau,\xi,\lambda)=t^\mu\tilde{g}_{\frac{1}{2}-\gamma_p}(\tau,\xi,t^\mu\tfrac{\lambda}{[\lambda]}),$$

is uniformly bounded in $L_p(\mathbb{R}_+\times\mathbb{R}^n,t^n dtdx)$ for large $|\lambda|$. Since then $\frac{\lambda}{[\lambda]}$ is bounded away from zero and infinity, a simple calculation shows that

$$|(t\partial_t)^l\partial_\tau^k\partial_\xi^\alpha g'_{\frac{1}{2}-\gamma_p}(t,\tau,\xi,\lambda)|\le c_{kl\alpha}(1+|\tau|+|\xi|)^{-k-|\alpha|}$$

uniformly in (t,τ,ξ,λ), i.e., $g'(\lambda)\in MS^0(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{n+1}{2}-\gamma_p}\times\mathbb{R}^n)$ uniformly in λ. Then the result follows, see the end of Section 9. □

There are certain relations between A and g from i) respectively $P(\lambda)$ from ii), which we are going to study now.

Let $U\subset\mathbb{R}^n$ be a coordinate neighborhood for X, whose closure is contained in another coordinate neighborhood. Condition (E1) ensures that the local symbol

$$\tilde{q}^{(\mu)}(t,x,\tau,\xi):=t^\mu\sigma_\psi^\mu(A)(t,x,t^{-1}\tau,\xi),\tag{4.4}$$

cf. (3.4), exists up to $t=0$ and does not take values in Λ_Δ for all $(t,x)\in[0,1]\times\overline{U}$ and all $(\tau,\xi)\neq 0$.

LEMMA 4.8. *There exists a zero excision function χ on $\mathbb{R}$, such that for any $\varphi\in\mathcal{C}^\infty_{\mathrm{comp}}(U)$ and any $\sigma\in\mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$*

$$\varphi(x)\sigma(t)\chi(|\tau,\xi|^2+|\lambda|^{\frac{2}{\mu}})\big(\lambda-\tilde{q}^{(\mu)}(t,x,\tau,\xi)\big)^{-1}\;\in\;S^{-\mu,\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^n\times\mathbb{R}^{1+n};\Lambda).$$

PROOF. Let a denote the symbol in question. For shortness let us write $y=(t,x)$ and $\eta=(\tau,\xi)$. Since the (eigen)values of $\tilde{q}^{(\mu)}(y,\eta)$ are proportional to $|\eta|^\mu$ (uniformly for $y\in[0,1]\times\overline{U}$) and do not lie in Λ_Δ, there exists a constant $c>0$ such that $(\lambda-\tilde{q}^{(\mu)}(y,\eta))^{-1}$ is a smooth function in

$$\{(y,\eta,\lambda)\in[0,1]\times\overline{U}\times\mathbb{R}^{1+n}\times\mathbb{C}\mid\lambda\in\Lambda_\Delta\text{ or }|\lambda|\le c|\eta|^\mu\}.$$

Thus, if we choose χ in such a way that $\chi(|\eta|^2+|\lambda|^{\frac{2}{\mu}})$ vanishes for $|\eta|\le(\delta/c)^{\frac{1}{\mu}}$ and $|\lambda|\le\delta$, then a is smooth on $\overline{\mathbb{R}}_+\times\mathbb{R}^n\times\mathbb{R}^{1+n}\times\Lambda$ for $\Lambda=\Lambda(\delta,\theta)$. To verify that a is a symbol, it suffices to show that

$$|a(y,\eta,\lambda)|\le c\,(1+|\eta|^2+|\lambda|^{\frac{2}{\mu}})^{\frac{-\mu}{2}}\tag{4.5}$$

uniformly in $y\in[0,1]\times\overline{U}$ and $(\eta,\lambda)\in\mathbb{R}^{1+n}\times\Lambda$. Since a is anisotropic homogeneous of order $(-\mu,\mu)$ for large $(\eta,\lambda)\in\mathbb{R}^{1+n}\times\Lambda_\Delta$, estimate (4.5) holds on $\mathbb{R}^{1+n}\times\Lambda_\Delta$. It also holds for $|\lambda|\le\delta$ and $|\eta|$ sufficiently large, since then $|(\lambda-\tilde{q}^{(\mu)}(y,\eta))^{-1}|$ is $O(|\eta|^{-\mu})$ due to the above described behavior of the (eigen)values. For $|\lambda|$ and $|\eta|$ simultaneously small, estimate (4.5) holds anyway. □

For every $\beta\in\mathbb{R}$, we can associate with $\tilde{g}$ from Theorem 4.6.i) a local symbol

$$\tilde{g}_\beta=\tilde{g}_\beta(t,x,\tau,\xi,\lambda)\;\in\;S^{-\mu,\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^n\times\mathbb{R}^{1+n};\Lambda).$$

It is a consequence of the above mentioned parametrix construction in the cone calculus that the principal symbol of $\tilde{g}$ is determined by the inverted principal symbol of $\lambda - A$. With the notation from Lemma 4.8 we indeed have

$$\varphi(x)\sigma(t)\left\{\tilde{g}_\beta - \chi(|\tau,\xi|^2 + |\lambda|^{\frac{2}{\mu}})\big(\lambda - \tilde{q}^{(\mu)}\big)^{-1}\right\} \in S^{-\mu-1,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n};\Lambda). \tag{4.6}$$

Similarly, the local symbols of $P(\lambda)$ from Theorem 4.6.ii) can be approximated modulo $S^{-\mu-1,\mu}$ in terms of the inverted local principal symbol of $\lambda - A$.

5. Complex powers of cone differential operators

The aim of this section is to show that a cone differential operator A satisfying (E1), (E2), also satisfies condition (A3). More precisely we shall show:

Theorem 5.1. *Let A be elliptic with respect to $\gamma + \mu$ and Λ_Δ, having no spectrum in the keyhole $\Lambda = \Lambda(\delta,\theta)$, except perhaps 0. Then one can define A^z as in (2.1) and there exists a constant $c_p \geq 0$ such that for all $z \in \mathbb{H}$ with $|\mathrm{Re}\, z|$ sufficiently small*

$$\|A^z\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p\, e^{\theta|\mathrm{Im}\, z|}. \tag{5.1}$$

Let us first give a short outline of the proof. In view of Theorem 4.6 we can replace in (2.1) the resolvent $(\lambda - A)^{-1}$ by the right-hand side of (4.3). Then one obtains four integrals (corresponding to the four terms on the right-hand side of (4.3)), each of which has to be estimated as in (5.1). For the one associated with $G_\infty(\lambda)$ this is certainly possible, since $G_\infty(\lambda)$ is rapidly decreasing in λ and therefore

$$\Big\| \int_{\mathcal{C}} \lambda^z G_\infty(\lambda)\, d\lambda \Big\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p'\, \delta^{\mathrm{Re}\, z} e^{\theta|\mathrm{Im}\, z|}.$$

Obviously $\delta^{\mathrm{Re}\, z}$ is uniformly bounded for small $|\mathrm{Re}\, z|$.

For the integral connected with $(1-\sigma)P(\lambda)(1-\sigma_1)$ one can proceed exactly as in the proof of Theorem 1 of [**19**], since this term is localized away from the boundary, and there, $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ coincides with the usual L_p-spaces (note also the remark after formula (4.6)).

Hence it remains to consider the expressions

$$\sigma \int_{\mathcal{C}} \lambda^z G(\lambda)\, d\lambda\, \sigma_0, \qquad \sigma \int_{\mathcal{C}} \lambda^z t^\mu \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)(\lambda)\, d\lambda\, \sigma_0. \tag{5.2}$$

We shall start with the analysis of the first term. Letting

$$\mathcal{H}_p^{0,\gamma}(X^\wedge) = L_p(X^\wedge, t^{(\frac{n+1}{2}-\gamma)p}\tfrac{dt}{t}dx) = t^\gamma L_p(X^\wedge, t^{\frac{n+1}{2}p}\tfrac{dt}{t}dx),$$

it is obvious from Definition 3.1 that multiplication with any cut-off function $\sigma \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ induces continuous operators $\mathcal{H}_p^{0,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(X^\wedge)$ and $\mathcal{H}_p^{0,\gamma}(X^\wedge) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$. Estimating the first term in 5.2 thus reduces to the following proposition:

Proposition 5.2. *Let $G(\lambda) \in R_G^{-\mu,\mu}(X^\wedge;\Lambda,\gamma)$ and $G_z = \int_{\mathcal{C}} \lambda^z G(\lambda)\, d\lambda$ for $z \in \mathbb{H}$. Then $G_z \in \mathcal{L}(\mathcal{H}_p^{0,\gamma}(X^\wedge))$, and there exists a constant $c_p \geq 0$ such that for $|\mathrm{Re}\, z|$ sufficiently small*

$$\|G_z\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(X^\wedge))} \leq c_p\, e^{\theta|\mathrm{Im}\, z|}.$$

Proof. By conjugation with t^γ we can assume that $\gamma = 0$ (cf. the proof of Proposition 4.7). If we split the integral into three terms according to the decomposition of $\mathcal{C}$ in (9.1), the integral over $\mathcal{C}_2$ can be estimated in the desired way, since $\|G(\lambda)\|$ is bounded on $\mathcal{C}_2$. By symmetry, $\mathcal{C}_1$ and $\mathcal{C}_3$ can be treated in the same way. So we shall assume for the rest of the proof that

$$\mathcal{C}(t) = \mathcal{C}_1(t) = te^{i\theta}, \qquad -\infty < t \leq 1,$$

(for notational convenience we replace δ by 1). Also for convenience we suppress the x-variables from the notation. We shall frequently make use of the fact that, substituting $\lambda = \varrho^\mu e^{i\theta}$, we have

$$\int_{\mathcal{C}} f(\lambda)\,d\lambda = \mu e^{i\theta}\int_1^\infty f(\varrho^\mu e^{i\theta})\varrho^{\mu-1}\,d\varrho.$$

According to Definition 4.3, $G(\lambda)$ for $|\lambda|\ge 1$ is an integral operator (with respect to the scalar product in $\mathcal{H}_2^{0,0}(X^\wedge)$) with kernel

$$k(\lambda,t,s) = |\lambda|^{\frac{n+1}{\mu}}\tilde{k}(\lambda,|\lambda|^{\frac{1}{\mu}}t,|\lambda|^{\frac{1}{\mu}}s),$$

where, for some $\varepsilon>0$,

$$\tilde{k}(\lambda,t,s)\in S^{-1}(\Lambda)\,\widehat{\otimes}_\pi\,\mathcal{S}_0^\varepsilon(X^\wedge)\,\widehat{\otimes}_\pi\,\mathcal{S}_0^\varepsilon(X^\wedge)$$

(the fact that $\tilde{k}$ is classical will not play a role for the following calculations). Then G_z is an integral operator with kernel

$$k_z(t,s) = \int_{\mathcal{C}}\lambda^z k(\lambda,t,s)\,d\lambda. \tag{5.3}$$

Writing $\tilde{k}(\lambda,t,s) = (\tilde{\chi}(t)+(1-\tilde{\chi})(t))\tilde{k}(\lambda,t,s)(\tilde{\chi}(s)+(1-\tilde{\chi})(s))$ with the characteristic function $\tilde{\chi}$ of $[0,1]$, the proposition will be true, if we can show that in any of the four cases

$$k_z(t,s) = e^{-\theta|\mathrm{Im}\,z|}\int_{\mathcal{C}}\lambda^z\tilde{\chi}(|\lambda|^{\frac{1}{\mu}}t)k(\lambda,t,s)\tilde{\chi}(|\lambda|^{\frac{1}{\mu}}s)\,d\lambda \tag{5.4}$$

$$k_z(t,s) = e^{-\theta|\mathrm{Im}\,z|}\int_{\mathcal{C}}\lambda^z\tilde{\chi}(|\lambda|^{\frac{1}{\mu}}t)k(\lambda,t,s)(1-\tilde{\chi})(|\lambda|^{\frac{1}{\mu}}s)\,d\lambda \tag{5.5}$$

$$k_z(t,s) = e^{-\theta|\mathrm{Im}\,z|}\int_{\mathcal{C}}\lambda^z(1-\tilde{\chi})(|\lambda|^{\frac{1}{\mu}}t)k(\lambda,t,s)\tilde{\chi}(|\lambda|^{\frac{1}{\mu}}s)\,d\lambda \tag{5.6}$$

$$k_z(t,s) = e^{-\theta|\mathrm{Im}\,z|}\int_{\mathcal{C}}\lambda^z(1-\tilde{\chi})(|\lambda|^{\frac{1}{\mu}}t)k(\lambda,t,s)(1-\tilde{\chi})(|\lambda|^{\frac{1}{\mu}}s)\,d\lambda \tag{5.7}$$

the associated integral operators are bounded in $\mathcal{H}_p^{0,0}(X^\wedge)$, uniformly in $-\alpha\le\mathrm{Re}\,z<0$ for some $\alpha>0$. The cases (5.5) and (5.6) are equivalent by symmetry (i.e. passing to the adjoint). The proofs of all cases (5.4), (5.5), and (5.7) rely on the following Hardy inequalities:

$$\int_0^\infty\Big(\int_0^t g(s)\,ds\Big)^p t^{-1-r}\,dt \le \left(\tfrac{p}{r}\right)^p\int_0^\infty g(t)^p t^{p-1-r}\,dt \tag{5.8}$$

$$\int_0^\infty\Big(\int_t^\infty g(s)\,ds\Big)^p t^{-1+r}\,dt \le \left(\tfrac{p}{r}\right)^p\int_0^\infty g(t)^p t^{p-1+r}\,dt \tag{5.9}$$

for any non-negative function g on $\mathbb{R}_+$ and $r>0$ (cf. [**23**], Lemma 3.14, page 196). To begin with case (5.4) we use the fact that, for some fixed $\epsilon>0$,

$$|\tilde{k}(\lambda,t,s)|\le c|\lambda|^{-1}t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}+\varepsilon}$$

uniformly in $\lambda\in\mathcal{C}$ and $t,s>0$, to obtain

$$\begin{aligned}|k_z(t,s)| &\le c\,t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}+\varepsilon}\int_1^\infty\varrho^{\mu\mathrm{Re}\,z-1+2\varepsilon}\tilde{\chi}(\varrho t)\tilde{\chi}(\varrho s)\,d\varrho\\ &= \tfrac{c}{\mu\mathrm{Re}\,z+2\varepsilon}\left(\min(\tfrac{1}{t},\tfrac{1}{s})^{\mu\mathrm{Re}\,z+2\varepsilon}-1\right)\tilde{\chi}(t)\tilde{\chi}(s)t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}+\varepsilon}.\end{aligned}$$

Since $\mu\mathrm{Re}\,z$ is negative, the factor $\min(\frac{1}{t},\frac{1}{s})^{\mu\mathrm{Re}\,z}$ is uniformly bounded by 1 for $0<s,t\le 1$. If $-\frac{\varepsilon}{\mu}=-\alpha\le\mathrm{Re}\,z<0$ the factor $\frac{c}{\mu\mathrm{Re}\,z+2\varepsilon}$ can be estimated from above by a constant uniformly in $0<s,t\le1$. Since furthermore the kernel function $\tilde{\chi}(t)\tilde{\chi}(s)t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}+\varepsilon}$ belongs to $\mathcal{H}_p^{0,0}(X^\wedge)\otimes$

$\mathcal{H}^{0,0}_{p'}(X^\wedge)$ and thus induces a continuous operator in $\mathcal{H}^{0,0}_{p}(X^\wedge)$, it remains to consider the kernel $t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}+\varepsilon}\min(\frac{1}{t},\frac{1}{s})^{2\varepsilon}$. Because this kernel is symmetric in s and t, indeed it suffices to treat

$$k(t,s)=\begin{cases} t^{-\frac{n+1}{2}-\varepsilon}s^{-\frac{n+1}{2}+\varepsilon} & : s\le t\\ 0 & : s>t\end{cases}.$$

If G denotes the associated integral operator, then

$$\begin{aligned}\|Gu\|^p_{\mathcal{H}^{0,0}_p(X^\wedge)} &\le \int_0^\infty\Big(\int_0^\infty k(t,s)|u(s)|\,s^n\,ds\Big)^p t^{\frac{n+1}{2}p-1}\,dt=\int_0^\infty\Big(\int_0^t s^{\frac{n-1}{2}+\varepsilon}|u(s)|\,ds\Big)^p t^{-1-p\varepsilon}\,dt\\ &\le \Big(\tfrac{p}{p\varepsilon}\Big)^p\int_0^\infty |u(t)|^p\,t^{\frac{n+1}{2}p-1}dt=\big(\tfrac{1}{\varepsilon}\big)^p\|u\|^p_{\mathcal{H}^{0,0}_p(X^\wedge)}\end{aligned}$$

by Hardy's inequality (5.8). This finishes case (5.4). For case (5.5) observe that

$$|\tilde{k}(\lambda,t,s)(1-\tilde{\chi})(s)|\le c_N|\lambda|^{-1}t^{-\frac{n+1}{2}+\varepsilon}s^{-N}$$

for any $N\in\mathbb{N}$ uniformly in $\lambda\in\mathcal{C}$ and $s,t>0$. Then

$$|k_z(t,s)|\le c_N t^{-\frac{n+1}{2}+\varepsilon}s^{-N}\int_1^\infty \varrho^{\mu\mathrm{Re}\,z+\frac{n-1}{2}+\varepsilon-N}\tilde{\chi}(\varrho t)(1-\tilde{\chi})(\varrho s)\,d\varrho.$$

This expression equals zero if $s\le t$ and for $s>t$ we can estimate

$$|k_z(t,s)|\le c_N t^{-\frac{n+1}{2}+\varepsilon}s^{-N}\tilde{\chi}(t)\int_{1/s}^{1/t}\varrho^{\mu\mathrm{Re}\,z+\frac{n-1}{2}+\varepsilon-N}\,d\varrho=\tfrac{c_N}{\mu\mathrm{Re}\,z+\frac{n+1}{2}+\varepsilon-N}(k^1_z(t,s)-k^2_z(t,s))$$

with kernel functions k^1_z and k^2_z given by

$$k^1_z(t,s)=\tilde{\chi}(t)\begin{cases}0 & : s\le t\\ \big(\frac{t}{s}\big)^N t^{-n-1-\mu\mathrm{Re}\,z} & : s>t\end{cases},\qquad k^2_z(t,s)=\tilde{\chi}(t)\begin{cases}0 & : s\le t\\ (ts)^{-\frac{n+1}{2}}\big(\frac{t}{s}\big)^\varepsilon s^{-\mu\mathrm{Re}\,z} & : s>t\end{cases}.$$

In order to check the uniform boundedness of the integral operator K^1_z associated with k^1_z, $-\alpha\le\mathrm{Re}\,z<0$, on $\mathcal{H}^{0,0}_p(X^\wedge)$ we observe that

$$\mathcal{H}^{0,0}_p(X^\wedge)=t^\beta L_p(X^\wedge,t^n dtdx)$$

with $\beta=p(n+1)(\frac{1}{2}-\frac{1}{p})$. The boundedness of K^1_z is equivalent to the boundedness of $t^\beta K^1_z t^{-\beta}$ on $L_p(X^\wedge,t^n dtdx)$. To show it, we employ Schur's lemma: if N is sufficiently large, then

$$\int_0^\infty t^\beta k^1_z(t,s)\,s^{-\beta}\,s^n ds=\tilde{\chi}(t)\,t^{\beta+N-n-1-\mu\mathrm{Re}\,z}\int_t^\infty s^{-N+n-\beta}\,ds=\tfrac{1}{N-n+\beta-1}t^{-\mu\mathrm{Re}\,z}\tilde{\chi}(t)\le 1$$

and

$$\int_0^\infty t^\beta k^1_z(t,s)s^{-\beta}\,t^n dt=s^{-N-\beta}\int_0^{\min(1,s)}t^{\beta+N-1-\mu\mathrm{Re}\,z}\,dt=\tfrac{s^{-N-\beta}}{N-\mu\mathrm{Re}\,z+\beta}\min(1,s)^{\beta+N-\mu\mathrm{Re}\,z}\le 1.$$

To handle k^2_z first observe that we can drop the factor $s^{-\mu\mathrm{Re}\,z}$, since this is uniformly bounded by 1 in $s\le 1$ and $\mathrm{Re}\,z<0$, and if $s\ge 1$ and $-\frac{\varepsilon}{2\mu}\le\mathrm{Re}\,z<0$ then $\big(\frac{t}{s}\big)^{\varepsilon/2}s^{-\mu\mathrm{Re}\,z}\le 1$ for $0\le t\le 1$ (for $t\ge 1$ anyway $k^2_z(t,s)=0$). Thus we can assume that

$$k^2_z(t,s)=k^2(t,s)=\tilde{\chi}(t)\begin{cases}0 & : s\le t\\ (ts)^{-\frac{n+1}{2}}\big(\frac{t}{s}\big)^\varepsilon=t^{-\frac{n+1}{2}+\varepsilon}s^{-\frac{n+1}{2}-\varepsilon} & : s>t\end{cases}.$$

But then Hardy's inequality (5.9) shows that the integral operator associated with the kernel k^2_z is continuous in $\mathcal{H}^{0,0}_p(X^\wedge)$ with operator norm bounded by $\frac{1}{\varepsilon}$. This finishes case (5.5). For the final case (5.7) we use that

$$|(1-\tilde{\chi})(t)\tilde{k}(\lambda,t,s)(1-\tilde{\chi})(s)|\le c_N|\lambda|^{-1}t^{-N}s^{-N}$$

for any $N \in \mathbb{N}$ uniformly in $\lambda \in \mathcal{C}$ and $s, t > 0$. Then

$$\begin{aligned} |k_z(t,s)| &\le c_N t^{-N} s^{-N} \int_1^\infty \varrho^{\mu \mathrm{Re}\, z + n - 2N} (1-\tilde\chi)(\varrho t)(1-\tilde\chi)(\varrho s)\, d\varrho \\ &= -\tfrac{c_N}{\mu \mathrm{Re}\, z + n + 1 - 2N} t^{-N} s^{-N} \max\left(1, \tfrac{1}{t}, \tfrac{1}{s}\right)^{\mu \mathrm{Re}\, z + n + 1 - 2N}. \end{aligned}$$

The factor in front is obviously uniformly bounded in $\mathrm{Re}\, z < 0$ for N sufficiently large. Since $\mu \mathrm{Re}\, z$ is negative,

$$\begin{aligned} |(1-\tilde\chi)(t) k_z(t,s)(1-\tilde\chi)(s)| &\le c\,(1-\tilde\chi)(t) t^{-N} s^{-N} (1-\tilde\chi)(s), \\ |\tilde\chi(t) k_z(t,s)(1-\tilde\chi)(s)| &\le c\,\tilde\chi(t) t^{N-n-1} s^{-N} (1-\tilde\chi)(s), \\ |(1-\tilde\chi)(t) k_z(t,s)\tilde\chi(s)| &\le c\,(1-\tilde\chi)(t) t^{-N} s^{N-n-1} \tilde\chi(s). \end{aligned}$$

All these kernel functions belong to $\mathcal{H}_p^{0,0}(X^\wedge) \otimes \mathcal{H}_{p'}^{0,0}(X^\wedge)$ for sufficiently large N and thus induce continuous operators in $\mathcal{H}_p^{0,0}(X^\wedge)$. Hence it remains to investigate $\tilde\chi(t) k_z(t,s)\tilde\chi(s)$ and by symmetry even

$$k(t,s) = \begin{cases} 0 & : s \le t \\ s^{-N} t^{N-n-1} & : s > t \end{cases}.$$

Again Hardy's inequality (5.9) shows that the associated operator is $\mathcal{H}^{0,0}(X^\wedge)$-continuous. $\square$

We consider now the second term in (5.2). Using a partition of unity on X with any two functions supported in a single coordinate neighborhood, we can assume $X = \mathbb{R}^n$ and use local symbols compactly supported in x. To complete the proof of theorem (5.1) we make use of the decomposition (4.6) and of the fact (see Section 9) that operators defined by means of symbols $a \in MS^0(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$ are bounded on $\mathcal{H}_p^{0,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$, with norm estimated in terms of the seminorms associated to a. We treat the homogeneous principal symbol of g and the lower order part separately.

LEMMA 5.3. *Let $\tilde b \in S^{-\mu-1,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n; \Lambda)$ be compactly supported in t and*

$$b_z(t, x, \tfrac{n+1}{2} - \gamma + i\tau, \xi) = t^\mu \int_{\mathcal{C}} \lambda^z \tilde b(t, x, \tfrac{n+1}{2} - \gamma + i\tau, \xi, t^\mu \lambda)\, d\lambda.$$

For $\mathrm{Re}\, z < 0$ this defines a symbol $b_z \in MS^0(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$, and the symbol estimates of $e^{-\theta|\mathrm{Im}\, z|} b_z$ are uniform in $-1 \le \mathrm{Re}\, z < 0$. Consequently,

$$\|\mathrm{op}_M^{\gamma - \frac{n}{2}}(b_z)\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n))} \le c_p\, e^{\theta|\mathrm{Im}\, z|}$$

uniformly in $-1 \le \mathrm{Re}\, z < 0$.

PROOF. Without loss of generality, we can set $\gamma = \frac{n+1}{2}$. We have to show that

$$|\partial_\tau^l (t\partial_t)^k \partial_\xi^\alpha \partial_x^\beta b_z(t, x, i\tau, \xi)| e^{-\theta|\mathrm{Im}\, z|} \langle \tau, \xi \rangle^{l+|\alpha|} \tag{5.10}$$

is uniformly bounded for $t > 0$, $x \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $-1 \le \mathrm{Re}\, z < 0$. The totally characteristic derivatives in t can be handled very simply, observing that $t\partial_t t^\mu = \mu t^\mu$,

$$t\partial_t \left(\tilde b(t, x, i\tau, \xi, t^\mu \lambda) \right) = (t\partial_t \tilde b)(t, x, i\tau, \xi, t^\mu\lambda) + \mu(\lambda\partial_\lambda \tilde b)(t, x, i\tau, \xi, t^\mu\lambda)$$

and both symbols $t\partial_t \tilde b$ and $\lambda \partial_\lambda \tilde b$ are of the same type as $\tilde b$. Since the derivatives with respect to x, τ and ξ can be taken under the integral sign, it suffices to assume $\tilde b \in S^{-\mu-1-k,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Gamma_0 \times \mathbb{R}^n; \Lambda)$ and to show that

$$|b_z(t, x, i\tau, \xi)| \le c\, e^{\theta|\mathrm{Im}\, z|} \langle \tau, \xi \rangle^{-k}$$

uniformly in $t > 0$, $x \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $-1 \le \mathrm{Re}\, z < 0$. By hypothesis, we have

$$|b_z(t, x, i\tau, \xi)| \le c\, t^\mu \int_{\mathcal{C}} |\lambda^z| (1 + \tau^2 + |\xi|^2 + |t^\mu \lambda|^{2/\mu})^{(-\mu-1-k)/2}\, d\lambda,$$

and on $\mathcal{C}$ we can estimate $|\lambda^z|$ from above by $\delta^{\mathrm{Re}\,z} e^{\theta|\mathrm{Im}\,z|}$. The transformation $\varrho = t^\mu\lambda$ yields

$$|b_z(t,x,i\tau,\xi)| \le c\,\delta^{\mathrm{Re}\,z} e^{\theta|\mathrm{Im}\,z|}\langle\tau,\xi\rangle^{-k}\int_{t^\mu\mathcal{C}} (1+|\varrho|)^{-1-\frac{1}{\mu}}\,d\varrho,$$

where $t^\mu\mathcal{C}$ means the path $\mathcal{C}(t^\mu\delta,\theta)$. Since the support of $\tilde{b}$ is compact, we may assume without loss of generality that $0<t\le 1$. Then we obtain the estimate

$$\int_{t^\mu\mathcal{C}} (1+|\varrho|)^{-1-\frac{1}{\mu}}\,d\varrho \le 2\pi\delta + \int_{\mathcal{C}_\Delta} (1+|\varrho|)^{-1-\frac{1}{\mu}}\,d\varrho < +\infty,$$

and the statement follows, since $\delta^{\mathrm{Re}\,z}$ is uniformly bounded in $-1\le \mathrm{Re}\,z<0$. □

Proposition 5.4. *Let $\tilde{g} = \tilde{g}(t,x,z,\xi,\lambda)$ be a local symbol associated to the Mellin symbol $\tilde{g}$ of $(\lambda-A)^{-1}$ of Theorem 4.6.i) and let*

$$g_z(t,x,\tfrac{n+1}{2}-\gamma+i\tau,\xi) = \sigma(t)t^\mu\int_{\mathcal{C}} \lambda^z\tilde{g}(t,x,\tfrac{n+1}{2}-\gamma+i\tau,\xi,t^\mu\lambda)\,d\lambda$$

with some cut-off function $\sigma\in\mathcal{C}^\infty_{comp}([0,1[)$. For $\mathrm{Re}\,z<0$ this defines a symbol $g_z\in MS^0(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{n+1}{2}-\gamma}\times\mathbb{R}^n)$, and the symbol estimates of $e^{-\theta|\mathrm{Im}\,z|}g_z$ are uniform in $-1\le\mathrm{Re}\,z<0$. In particular,

$$\|\mathrm{op}_M^{\gamma-\frac{n}{2}}(g_z)\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{R}_+\times\mathbb{R}^n))} \le c_p\,e^{\theta|\mathrm{Im}\,z|}$$

uniformly in $-1<\mathrm{Re}\,z<0$.

Proof. Without loss of generality let $\gamma=\frac{n+1}{2}$. We shall also suppress σ from the notation and instead assume that $0<t\le1$. We can also assume x confined to a compact subset of $\mathbb{R}^n$. By (4.6),

$$\tilde{g}(t,x,i\tau,\xi,\lambda) = \chi(|\tau,\xi|^2+|\lambda|^{\frac{2}{\mu}})(\lambda-\tilde{q}^{(\mu)}(t,x,i\tau,\xi))^{-1} \mod S^{-\mu-1,\mu}(\overline{\mathbb{R}}_+\times\mathbb{R}^n\times\mathbb{R}^{1+n};\Lambda),$$

where $\tilde{q}^{(\mu)}$ denotes a local symbol of A as defined in (4.4). In view of Lemma 5.3, we therefore may assume that

$$\begin{aligned} g_z(t,x,i\tau,\xi) &= t^\mu\int_{\mathcal{C}}\lambda^z\chi(|\tau,\xi|^2+t^2|\lambda|^{\frac{2}{\mu}})(t^\mu\lambda-\tilde{q}^{(\mu)}(t,x,i\tau,\xi))^{-1}\,d\lambda \\ &= t^{-\mu z}\int_{t^\mu\mathcal{C}}\varrho^z\chi(|\tau,\xi|^2+|\varrho|^{\frac{2}{\mu}})(\varrho-\tilde{q}^{(\mu)}(t,x,i\tau,\xi))^{-1}\,d\varrho, \end{aligned}$$

where we have used the substitution $\varrho=t^\mu\lambda$. We have to estimate this expression as in (5.10). The factor $t^{-\mu z}$ behaves correctly, since $(t\partial_t)^k t^{-\mu z} = (-\mu z)^k t^{-\mu z}$ is uniformly bounded in $0<t\le1$ and $-1\le\mathrm{Re}\,z<0$. For $(\tau,\xi)\ne0$ we have

$$\mathrm{spec}(\tilde{q}^{(\mu)}(t,x,i\tau,\xi)) \subset \{\lambda\in\mathbb{C} \mid c_1|\tau,\xi|^\mu \le |\lambda| \le c_2|\tau,\xi|^\mu \text{ and } |\arg\lambda|<\theta\}$$

with suitable constants c_1 and c_2. Thus for large enough $|\tau,\xi|$ we have $\chi(|\tau,\xi|^2+t^{-2\mu}|\varrho|^2)=1$ and the spectrum of $\tilde{q}^{(\mu)}(t,x,i\tau,\xi)$ is located to the right of the path $\mathcal{C}$. By Cauchy's theorem we can then replace the path $t^\mu\mathcal{C}$ by $\mathcal{C}$, and obtain for large $|\tau,\xi|$

$$\int_{\mathcal{C}}\varrho^z(\varrho-\tilde{q}^{(\mu)}(t,x,i\tau,\xi))^{-1}\,d\varrho = 2\pi i\,\tilde{q}^{(\mu)}(t,x,i\tau,\xi)^z.$$

Then we can estimate (as in **[19]**, (2.9))

$$|\partial^l_\tau(t\partial_t)^k\partial^\alpha_\xi\partial^\beta_x g_z(t,x,i\tau,\xi)| \le p(|z|)e^{\theta|\mathrm{Im}\,z|}\langle\tau,\xi\rangle^{\mu\mathrm{Re}\,z-l-|\alpha|} \le p(|z|)e^{\theta|\mathrm{Im}\,z|}\langle\tau,\xi\rangle^{-l-|\alpha|}$$

with a polynomial p. However, since we can replace θ by $\theta-\varepsilon$ for some $\varepsilon>0$ (as noted in the comments on conditions (E1) and (E2)), this yields the uniform symbol estimates of g_z for large $|\tau,\xi|$.

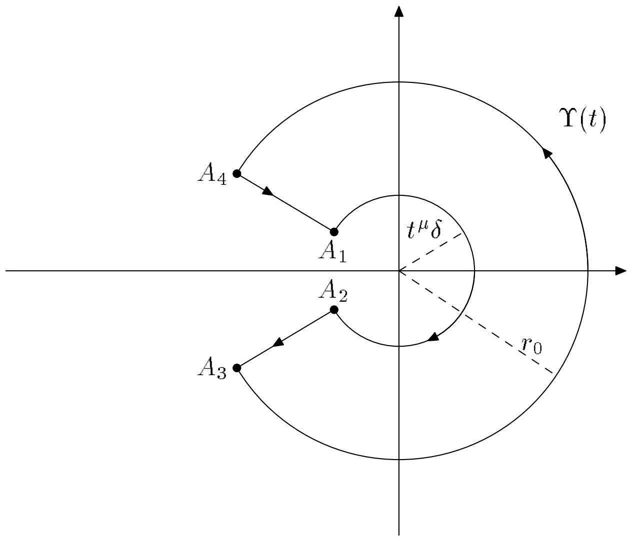
For small $|\tau, \xi|$, we now shall show that

$$g_z(t,x,i\tau,\xi) = t^{-\mu z} \int_{\Upsilon(t)} \varrho^z a(t,x,\tau,\xi;\varrho)\, d\varrho, \tag{5.11}$$

where we have set

$$a(t,x,\tau,\xi;\varrho) = \chi(|\tau,\xi|^2 + |\varrho|^{\frac{2}{\mu}})(\varrho - \tilde{q}^{(\mu)}(t,x,i\tau,\xi))^{-1} \in S^{-\mu,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n};\Lambda)$$

and $\Upsilon(t)$ is the path given in the following picture (with $r_0 > 0$ to be chosen appropriately):



In fact, the difference of both sides from (5.11) equals

$$\alpha(r) = \int_{\mathcal{C}(r,\theta)} \varrho^z a(t,x,\tau,\xi;\varrho)\, d\varrho \quad \text{for } r = r_0. \tag{5.12}$$

Since, for small $|\tau,\xi|$, the spectrum of $\tilde{q}^{(\mu)}(t,x,i\tau,\xi)$ is contained in some ball of finite radius, $a(t,x,\tau,\xi;\varrho)$ is holomorphic in ϱ for $|\varrho| \geq r_0$, if r_0 is chosen large enough. Thus $\alpha(r) = \alpha(r_0)$ for all $r \geq r_0$, by Cauchy's theorem. For any fixed z and (t,x,τ,ξ) the integrand in (5.12) is $O(|\varrho|^{-1+\mathrm{Re}\, z})$ for $|\varrho| \to \infty$ and, on the radial part of $\mathcal{C}(r)$, the integrand is $O(r^{\mathrm{Re}\, z})$. Hence, $\alpha(r_0) = \lim_{r\to+\infty} \alpha(r) = 0$, and (5.11) holds.

To estimate the right-hand side of (5.11), we split the integral into four parts, which we briefly analyze separately. First of all, observe that $|\varrho^z|$ can be estimated from above by $e^{\theta|\mathrm{Im}\, z|}(t^\mu\delta)^{\mathrm{Re}\, z}$ on the whole path. This and the fact that $a(t,x,\tau,\xi;\varrho) \in S^{-\mu,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n};\Lambda)$ are enough to get the desired estimates for the terms obtained integrating along the two arcs $\widehat{A_1A_2}$ and $\widehat{A_3A_4}$, since they can be treated with essentially the same technique we used to prove Lemma 5.3. The term obtained integrating along $\overline{A_2A_3}$ is

$$b(t,x,\tau,\xi) = \int_{t^\mu\delta}^{r_0} (se^{i\theta})^z a(t,x,\tau,\xi;se^{i\theta}) e^{i\theta}\, ds.$$

The derivatives with respect to x, ξ and τ can be taken under the integral sign, so that we could again start with a symbol $a \in S^{-\mu-k,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n};\Lambda)$ and prove that, for any l,

$$|(t\partial_t)^l b(t,x,\tau,\xi)| \leq c_l e^{\theta|\mathrm{Im}\, z|}\langle\tau,\xi\rangle^{-k}$$

uniformly in $-1 < \mathrm{Re}\, z < 0$. This is true for $l = 0$, as one can easily check. For $l = 1$ we get

$$t\partial_t b(t,x,\tau,\xi) = \int_{t^\mu\delta}^{r_0} (se^{i\theta})^z (t\partial_t a)(t,x,\tau,\xi;se^{i\theta}) e^{i\theta}\, ds - \mu(t^\mu\delta e^{i\theta})^{z+1} a(t,x,\tau,\xi;t^\mu\delta e^{i\theta}),$$

and this also satisfies the desired estimate. In fact, the first term is of the same kind as b, while for the second it suffices to use the definition of $S^{-\mu-k,\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n};\Lambda)$ and the fact that $0 < t \leq 1$. The result for arbitrary l can be proved by induction and, obviously, the contribution obtained integrating along $\overline{A_4A_1}$ behaves in a completely similar way. This yields the desired symbol estimates for small $|\tau,\xi|$ and finishes the proof. $\square$

REMARK 5.5. *Let us point out that the proof of Theorem* 5.1 *only makes use of assumption* (E1), *the structure of the resolvent* (4.6), *and the fact that* $\operatorname{spec} A \cap \Lambda = \emptyset$. *It does not use that the domain* $\mathcal{D}(A)$ *equals* $\mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B})$. *Therefore, Theorem* 5.1 *holds true for* A *considered on other domains, as long as* (E1), (4.6), *and* $\operatorname{spec} A \cap \Lambda = \emptyset$ *are satisfied.*

In the next section we shall investigate the possible closed extensions of A, and use the previous remark to obtain an analogue of Theorem 5.1 for the maximal extension of A.

6. Closed extensions of cone differential operators

Let A be a cone differential operator, which is elliptic with respect to $\gamma+\mu$ in the sense of Remark 3.2. If we consider A as the unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with domain $\mathcal{C}^\infty_{\text{comp}}(\operatorname{int}\mathbb{B})$, its closure $A_{\min} = A_{\min}^{\gamma,p}$ is given by

$$\mathcal{D}(A_{\min}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}),$$

and the maximal closed extension $A_{\max} = A_{\max}^{\gamma,p}$ by

$$\mathcal{D}(A_{\max}) = \left\{u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \mid Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})\right\}.$$

Note that in (3.8) we simply wrote A instead of $A_{\min}$. Taking into account the duality of $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ and $\mathcal{H}_{p'}^{0,-\gamma}(\mathbb{B})$, the following lemma is valid:

LEMMA 6.1. *If* A^t *is the formal adjoint of* A *with respect to the scalar product of* $\mathcal{H}_2^{0,0}(\mathbb{B})$, *then*

$$(A_{\min}^{\gamma,p})^* = (A^t)_{\max}^{-\gamma,p'}, \qquad (A_{\max}^{\gamma,p})^* = (A^t)_{\min}^{-\gamma,p'}.$$

We shall write this more shortly as $A_{\min}^* = A_{\max}^t$ *and* $A_{\max}^* = A_{\min}^t$.

A proof of the above statements in case $p = 2$ and $\gamma = 0$ is given in **[9]**. The argument in the general case is analogous. As a simple consequence,

$$(\lambda - A_{\max})^{-1} = [(\overline{\lambda} - A_{\min}^t)^{-1}]^*$$

whenever one of both sides exists. Since the structure of the resolvent of $A = A_{\min}$ as given in Theorem 4.6 is invariant under passing to the adjoint, we obtain the following theorem:

THEOREM 6.2. *If* $A_{\min}^t$ *is elliptic with respect to* Λ_Δ *and* $-\gamma+\mu$, *then* $A_{\max}$ *has no spectrum in* $\Lambda_\Delta \cap \{|\lambda| > R\}$ *for some* $R > 0$, *and for large* $\lambda \in \Lambda_\Delta$

$$(\lambda - A_{\max})^{-1} = \sigma\left\{t^\mu \operatorname{op}_M^{\gamma-\frac{n}{2}}(g)(\lambda) + G(\lambda)\right\}\sigma_0 + (1-\sigma)P(\lambda)(1-\sigma_1) + G_\infty(\lambda),$$

where $\sigma, \sigma_0, \sigma_1 \in \mathcal{C}^\infty_{\text{comp}}([0,1[)$ *are cut-off functions satisfying* $\sigma_1\sigma = \sigma_1$, $\sigma\sigma_0 = \sigma$, *and*

i) $g(t,z,\lambda) = \tilde{g}(t,z,t^\mu\lambda)$ *with* $\tilde{g} \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-\mu,\mu}(X;\Lambda))$,
ii) $P(\lambda) \in L^{-\mu,\mu}(\operatorname{int}\mathbb{B};\Lambda)$,
iii) $G(\lambda) \in R_G^{-\mu,\mu}(X^\wedge;\Lambda,\gamma)$, *and* $G_\infty(\lambda) \in C_G^{-\infty}(\mathbb{B};\Lambda,\gamma)$.

Proceeding exactly as in Proposition 4.7 and Section 5, we can prove a norm estimate for the complex powers of $A_{\max}$:

Theorem 6.3. *Let $A^t_{\min}$ be elliptic with respect to $-\gamma+\mu$ and Λ_Δ, having no spectrum in the keyhole $\Lambda=\Lambda(\delta,\theta)$ except perhaps 0. Then one can define $A^z_{\max}$ as in* (2.1) *and there exists a constant $c_p\ge 0$ such that for all $z\in\mathbb{H}$ with $|\mathrm{Re}\, z|$ sufficiently small*

$$\|A^z_{\max}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))}\le c_p\, e^{\theta|\mathrm{Im}\, z|}.$$

Of course, it is desirable to express the ellipticity assumptions made on $A^t_{\min}$ in the previous two theorems purely in terms of $A_{\max}$. This can be done as follows.

Remark 6.4. *If $\widehat{A}$ is the model cone operator on $X^\wedge$ associated with A and $\widehat{A}_{\max}=\widehat{A}^{\gamma,p}_{\max}$ is the closed operator given by*

$$\mathcal{D}(\widehat{A}_{\max})=\left\{u\in\mathcal{K}^{0,\gamma}_p(X^\wedge)\mid \widehat{A}u\in\mathcal{K}^{0,\gamma}_p(X^\wedge)\right\},$$

then $A^t_{\min}$ is elliptic with respect to $-\gamma+\mu$ and Λ_Δ if and only if A satisfies condition (E1) *and*

(E2$'$) *$\widehat{A}_{\max}$ has no spectrum in $\Lambda_\Delta\setminus\{0\}$.*

The previous remark holds true, since similar to Lemma 6.1, $\widehat{A}^*_{\min}=\widehat{A}^t_{\max}$ and $\widehat{A}^*_{\max}=\widehat{A}^t_{\min}$.

It can be shown that $\mathcal{D}(A_{\max})$ differs from $\mathcal{D}(A_{\min})$ by a finite dimensional space (for the case $p=2$ see [**9**]),

$$\mathcal{D}(A_{\max})=\mathcal{D}(A_{\min})\oplus V,\qquad \dim V<\infty.$$

More precisely, the dimension of V only depends on the conormal symbol of A,

$$\dim V=\sum_{-2<\mathrm{Re}\, z-\frac{n+1}{2}+\gamma<0} M(\sigma^\mu_M(A),z), \tag{6.1}$$

where $M(h,z)$ denotes the multiplicity in z in the sense of [**7**] of a function h, which is holomorphic in a punctured neighborhood of z. Moreover, V consists of smooth functions of the form

$$\omega(t)\sum_{j=0}^{N}\sum_{k=0}^{k_j}c_{jk}(x)t^{-p_j}(\log t)^k,\qquad c_{jk}\in\mathcal{C}^\infty(X);$$

the coefficients c_{jk}, the exponents $p_j\in\mathbb{C}$ ($\frac{n+1}{2}-\gamma-\mu\le\mathrm{Re}\, p_j<\frac{n+1}{2}-\gamma$), and $k_j,N\in\mathbb{N}_0$ are determined by A. In particular, the only closed extensions of A are the operators A_W given by

$$\mathcal{D}(A_W)=\mathcal{D}(A_{\min})\oplus W,\qquad W\le V.$$

In this notation, $A_{\min}=A_{\{0\}}$ and $A_{\max}=A_V$. Correspondingly,

$$\mathcal{D}(\widehat{A}_{\max})=\mathcal{D}(\widehat{A}_{\min})\oplus\widehat{V},\qquad \dim\widehat{V}=\dim V,$$

and all closed extensions $\widehat{A}_{\widehat{W}}$ are given by

$$\mathcal{D}(\widehat{A}_{\widehat{W}})=\mathcal{D}(\widehat{A}_{\min})\oplus\widehat{W},\qquad \widehat{W}\le\widehat{V}.$$

Remark 6.5. *If the conormal symbol $\sigma^\mu_M(A)(z)$ is invertible for all $\frac{n+1}{2}-\gamma-2<\mathrm{Re}\, z<\frac{n+1}{2}-\gamma$, then $\dim V=\dim\widehat{V}=0$ by* (6.1)*, and both A and $\widehat{A}$ have only one closed extension in $\mathcal{H}^{0,\gamma}_p(\mathbb{B})$ and $\mathcal{K}^{0,\gamma}_p(X^\wedge)$, respectively.*

7. Example: A Cauchy Problem for Laplacians

Let $g(t)$ be a family of metrics on X, depending smoothly on a parameter $t\in\overline{\mathbb{R}}_+$, and $\Delta_X(t)$ the corresponding Laplacian on X. If we equip $\mathrm{int}\,\mathbb{B}$ with a metric that coincides with $dt^2+t^2g(t)$ near $t=0$, the associated Laplacian Δ is near the boundary given by

$$t^{-2}\left\{(t\partial_t)^2+(n-1+tG^{-1}(t)(\partial_t G)(t))t\partial_t+\Delta_X(t)\right\},$$

where $G=(\det(g_{ij}))^{\frac12}$ and $n=\dim X$. Hence Δ is a cone differential operator in the sense of (3.1). We shall prove the following theorem:

THEOREM 7.1. *Let Δ be the Laplacian on* $\mathrm{int}\,\mathbb{B}$ *in the above sense,* $1<p<\infty$ *such that*

$$2\max(p,p')-1<n=\dim\partial\mathbb{B}. \tag{7.1}$$

If $\gamma_p=(n+1)(\frac{1}{2}-\frac{1}{p})$, *then Δ defined on* $\mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B})$ *has for any* $1<q<\infty$ *a unique closed extension* $\Delta_{p,q}$ *in* $\mathcal{H}^{0,\gamma_p}_q(\mathbb{B})$, *which is given by*

$$\mathcal{D}(\Delta_{p,q})=\mathcal{H}^{2,\gamma_p+2}_q(\mathbb{B}).$$

Moreover, $-\Delta_{p,q}$ *is elliptic with respect to* γ_p+2 *and any sector* $\Lambda_\Delta\subset\mathbb{C}\setminus\mathbb{R}_+$.

PROOF. Let us set $A=-\Delta$. The rescaled symbol of A is

$$\tilde{\sigma}^2_\psi(A)(x,\tau,\xi)=\tau^2+|\xi|_x,$$

where $|\xi|$ refers to the metric $g(0)$ on X. Hence A satisfies the ellipticity condition (E1) for any Λ_Δ in question. The conormal symbol of A, cf. (3.6) and (3.7), is

$$\sigma^2_M(A)(z)=-z^2+(n-1)z-\Delta_X(0):H^s(X)\longrightarrow H^{s-2}(X).$$

If $0=\lambda_0\geq\lambda_1\geq\dots$ are the eigenvalues of $\Delta_X(0)$, then $\sigma^2_M(A)(z)$ is not bijective if and only if

$$z\in\left\{\tfrac{n-1}{2}\pm\left(\tfrac{(n-1)^2}{4}-\lambda_j\right)^{\frac{1}{2}}\mid j\in\mathbb{N}_0\right\}.$$

Note that, in particular, $\sigma^2_M(A)(z)$ is invertible for all z with $0<\mathrm{Re}\,z<n-1$, and thus by condition (7.1) for all z with $\frac{n+1}{2}-\gamma_p-2\leq\mathrm{Re}\,z\leq\frac{n+1}{2}-\gamma_p$. This shows that A is elliptic with respect to γ_p+2 in the sense of Remark 3.2 and has only one closed extension

$$A_{p,q}:\mathcal{H}^{2,\gamma_p+2}_q(\mathbb{B})\subset\mathcal{H}^{0,\gamma_p}_q(\mathbb{B})\longrightarrow\mathcal{H}^{0,\gamma_p}_q(\mathbb{B})$$

by Remark 6.5. The model cone operator is

$$\widehat{A}=-t^{-2}\left\{(t\partial_t)^2+(n-1)(t\partial_t)+\Delta_X(0)\right\},$$

i.e. $-\widehat{A}$ is the Laplacian on $X^\wedge$ with respect to the metric $dt^2+t^2g(0)$. As before, $\widehat{A}$ has a unique closed extension

$$\widehat{A}_{p,q}:\mathcal{K}^{2,\gamma_p+2}_q(X^\wedge)\subset\mathcal{K}^{0,\gamma_p}_q(X^\wedge)\longrightarrow\mathcal{K}^{0,\gamma_p}_q(X^\wedge).$$

Since $\widehat{A}$ is symmetric and non-negative, $\widehat{A}_{2,2}$ is self-adjoint and $\mathrm{spec}(\widehat{A}_{2,2})\subset\overline{\mathbb{R}}_+$. Let us show that

$$\mathrm{spec}(\widehat{A}_{p,q})\subset\overline{\mathbb{R}}_+\qquad\forall\,1<q<\infty,\ p\text{ satisfying (7.1)}.$$

By Corollary 3.15 of [**14**] (in the version for operators in the cone algebra $C^\mu(X^\wedge;(\gamma,\gamma-\mu,\Theta))$ on $X^\wedge$, which is introduced in Section 8.2.5 of [**5**]), the spectrum of $\widehat{A}_{p,q}$ is independent of $1<q<\infty$. Thus we can set $q=2$ and write $\widehat{A}_p=\widehat{A}_{p,2}$. We can assume $p\geq 2$, by passing to the adjoint (i.e., $\widehat{A}^*_p=\widehat{A}_{p'}$ and $-\gamma_p=\gamma_{p'}$). Then $\ker(\lambda-\widehat{A}_p)\subset\ker(\lambda-\widehat{A}_2)=\{0\}$, since $\mathcal{K}^{2,\gamma_p+2}_2(X^\wedge)\subset\mathcal{K}^{2,2}_2(X^\wedge)$ in view of $\gamma_p\geq\gamma_2=0$. The fact that $\sigma^2_M(A)(z)$ is invertible for $0<\mathrm{Re}\,z\leq\frac{n+1}{2}-\gamma_{p'}-2$ implies that

$$\ker(\lambda-\widehat{A}_{p'})\subset\mathcal{K}^{2,2}_2(X^\wedge)=\mathcal{K}^{2,\gamma_2+2}_2(X^\wedge),\quad\lambda\notin\overline{\mathbb{R}}_+; \tag{7.2}$$

we shall give the argument below. As a consequence, we have for the adjoint

$$\ker(\lambda-\widehat{A}_p)^*=\ker(\bar{\lambda}-\widehat{A}_{p'})\subset\ker(\bar{\lambda}-\widehat{A}_2),\quad\lambda\notin\overline{\mathbb{R}}_+,$$

hence $\lambda-\widehat{A}_p$ is bijective for $\lambda\notin\overline{\mathbb{R}}_+$.

In order to see (7.2) set $\gamma^1=\min(\gamma_{p'}+2,0)$. The invertibility of the conormal symbol implies that $\lambda-\widehat{A}$ is elliptic with respect to γ^1+2. Moreover, the minimal and maximal extensions of $\lambda-\widehat{A}$, considered as unbounded operators in $\mathcal{K}^{2,\gamma^1}_2(X^\wedge)$, coincide and their domain is $\mathcal{K}^{2,\gamma^1+2}_2(X^\wedge)$. In particular, $\mathcal{N}=\ker\{\lambda-\widehat{A}:\mathcal{K}^{2,\gamma_{p'}+2}_2(X^\wedge)\to\mathcal{K}^{2,\gamma_{p'}}_2(X^\wedge)\}$ is a subset of the maximal domain, thus it is included in $\mathcal{K}^{2,\gamma^1+2}_2(X^\wedge)$. Iterating this process, we see that $\mathcal{N}\subset\mathcal{K}^{2,\gamma^j+2}_2(X^\wedge)$ for all $\gamma^j:=\min(\gamma^{j-1}+2,0)=\min(\gamma_{p'}+2j,0)$. Choosing j large enough we get (7.2). ∎

Let us mention that Δ is also considered in [**10**]. As a consequence of Theorem 7.1 we get the following result on the solvability of the Cauchy Problem for the Laplacian:

THEOREM 7.2. *Let Δ be the Laplacian on* $\operatorname{int}\mathbb{B}$ *as described above,* $1 < p < \infty$*, and* $2\max(p,p') < \dim\mathbb{B}$*. Then the Cauchy problem*

$$u'(t) - \Delta u(t) = f(t), \quad 0 \le t \le T; \qquad u(0) = 0, \tag{7.3}$$

has for any

$$f \in L_r\left([0,T], \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})\right), \qquad 1 < q, r < \infty,$$

a unique solution

$$u \in W_r^1\left([0,T], \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})\right) \cap L_r\left([0,T], \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})\right).$$

Furthermore, u, u', and Δu depend continuously on f.

In fact, in Theorem 7.1 above, we have shown that $-\Delta$ is elliptic with respect to γ_p+2 and any sector Λ_Δ not containing $\overline{\mathbb{R}}_+$. Moreover, the problem (7.3) is equivalent to $v'(t) - (\Delta - c)v(t) = e^{ct}f(t)$, $v(0) = 0$, and, for sufficiently large c, the operator $-\Delta + c$ satisfies the assumptions of Theorem 5.1 for any fixed $0 < \theta < \frac{\pi}{2}$ and $\delta > 0$. Then

$$\|(-\Delta + c)^{iy}\|_{\mathcal{L}(\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}))} \le c_{p,q}\, e^{\theta|y|} \qquad \forall\, y \in \mathbb{R},$$

and Theorem 7.2 immediately follows from Theorem 3.2 of [**3**].

8. Appendix: Smoothing Mellin symbols and Green symbols

The structure of the resolvent (respectively parametrix) of a differential operator A as given in Theorem 4.6 at the first glance does not coincide with those which you find for example in [**5**] or [**6**]. This is mainly due to the fact that we consider A as an unbounded operator in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ whose resolvent acts continuously in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, and do not consider A as a bounded operator acting from $\mathcal{H}_p^{s+\mu,\gamma+\mu}(\mathbb{B})$ to $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$. We shall use this appendix to clarify this point.

Let us begin with a discussion of so-called *Green symbols*. Let us set

$$\mathcal{K}^{s,\gamma}(X^\wedge)^\nu = \langle t\rangle^\nu\, \mathcal{K}_2^{s,\gamma}(X^\wedge)$$

for real ν, cf. Definition 3.4. These are Hilbert spaces, and $\mathcal{K}^{-s,-\gamma}(X^\wedge)^{-\nu}$ can be identified with the dual space of $\mathcal{K}^{s,\gamma}(X^\wedge)^\nu$ via the scalar-product in $\mathcal{K}^{0,0}(X^\wedge)$. The operators κ_ϱ defined in (3.11) extend by continuity to operators in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\nu)$.

For $\mu \in \mathbb{R}$ and $d > 0$ we let

$$S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'})$$

denote the space of all smooth functions $a \in \mathcal{C}^\infty(\Lambda, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'}))$ satisfying

$$\|\kappa_{\langle\lambda\rangle^{-1/d}}\{\partial_\lambda^\alpha a(\lambda)\}\kappa_{\langle\lambda\rangle^{1/d}}\| \le c_\alpha\, \langle\lambda\rangle^{\frac{\mu}{d}-|\alpha|}$$

uniformly for $\lambda \in \Lambda$ and all multiindices α.

We call a smooth function $b \in \mathcal{C}^\infty(\Lambda_\Delta \setminus 0, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'}))$ *twisted homogeneous* of degree (μ, d) if it fulfills

$$b(\varrho^d\lambda) = \varrho^\mu\, \kappa_\varrho\, b(\lambda)\, \kappa_\varrho^{-1}$$

for all λ and $\varrho > 0$. Note that multiplying b with a 0-excision function (supported sufficiently far away from zero) yields a symbol in $S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'})$. The space

$$S_{cl}^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'})$$

then consists of all symbols from $S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\nu'})$, that have asymptotic expansions $a \sim \sum_{k=0}^{\infty} a^{(\mu-j,d)}$ with functions $a^{(\mu-j,d)}$ that are twisted homogeneous of degree $(\mu - j, d)$.

DEFINITION 8.1. *Let $\gamma, \gamma' \in \mathbb{R}$. If $g \in S^{\mu,d}(\Lambda; \mathcal{K}^{0,\gamma}(X^\wedge), \mathcal{K}^{0,\gamma'}(X^\wedge))$, we can form the adjoint symbol $g^* \in S^{\mu,d}(\Lambda; \mathcal{K}^{0,-\gamma'}(X^\wedge), \mathcal{K}^{0,-\gamma}(X^\wedge))$ by taking pointwise the adjoint with respect to the $\mathcal{K}^{0,0}(X^\wedge)$-scalar product. We then call g a Green symbol if additionally there exists an $\varepsilon = \varepsilon(g) > 0$, such that*

$$g \in \bigcap_{s,s',\nu,\nu'} S^{\mu,d}_{cl}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma'+\varepsilon}(X^\wedge)^{\nu'}),$$
$$g^* \in \bigcap_{s,s',\nu,\nu'} S^{\mu,d}_{cl}(\Lambda; \mathcal{K}^{s,-\gamma'}(X^\wedge)^\nu, \mathcal{K}^{s',-\gamma+\varepsilon}(X^\wedge)^{\nu'}).$$

The entity of all such Green symbols we shall denote by

$$R^{\mu,d}_G(X^\wedge; \Lambda, (\gamma, \gamma')).$$

It is a trivial fact that if $\gamma' \geq \gamma''$, then

$$R^{\mu,d}_G(X^\wedge; \Lambda, (\gamma, \gamma')) \subset R^{\mu,d}_G(X^\wedge; \Lambda, (\gamma, \gamma'')).$$

Moreover it can be shown, cf. [**20**], that in case $\gamma = \gamma'$ both Definitions 8.1 and 4.3 yield the same symbols respectively operator-families. In other words, Green symbols can either be characterized by their mapping properties in Sobolev spaces or by the structure of their kernels.

Let us now return to the resolvent, cf. Theorem 4.6. If you compare with [**6**], you will find that there 'our' term $G(\lambda)$ is replaced by a term of the form $G_0(\lambda) + M(\lambda)$, where

$$G_0(\lambda) \in R^{-\mu,\mu}_G(X^\wedge; \Lambda, (\gamma, \gamma + \mu)),$$

is a Green symbol and

$$M(\lambda) = \omega(t[\lambda]^{\frac{1}{\mu}})\, t^\mu \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\, \omega_0(t[\lambda]^{\frac{1}{\mu}})$$

for some cut-off functions $\omega, \omega_0 \in \mathcal{C}^\infty_{\mathrm{comp}}(\overline{\mathbb{R}}_+)$, and a meromorphic Mellin symbol

$$h \in M_P^{-\infty}(X).$$

The last notation roughly means that h is a meromorphic function on the complex plane with values in $L^{-\infty}(X)$, the smoothing pseudodifferential operators on X, having only finitely many poles in any vertical strip $|\mathrm{Re}\, z| \leq \beta$, $\beta > 0$, and the Laurent coefficients of the principal part of h at any such pole are finite rank operators. For more details see [**5**], Section 8.1.2. By the above observation,

$$G_0(\lambda) \in R^{-\mu,\mu}_G(X^\wedge; \Lambda, \gamma).$$

The same is also true for M, since it is easy to see that

$$M \in S^{-\mu,\mu}_{cl}(\Lambda; \mathcal{K}^{s,\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',\gamma+\mu}(X^\wedge)^{\nu'})$$

for all s, s', ν, ν' (note that M is twisted homogeneous for large $|\lambda|$). Observe that $\mu \geq 1$ since we are dealing with differential operators of positive order. The adjoint symbol is given by

$$M^*(\lambda) = \omega_0(t[\lambda]^{\frac{1}{\mu}})\, t^\mu \operatorname{op}_M^{-\gamma_+ -\frac{n}{2}}(h^*)\, \omega(t[\lambda]^{\frac{1}{\mu}}) \; + \; G_1(\lambda)$$

where $h^*(z) = h(n + 1 - \mu - \overline{z})^*$ and

$$G_1(\lambda) = t^\mu\, \omega_0(t[\lambda]^{\frac{1}{\mu}}) \left\{ \operatorname{op}_M^{-\gamma-\mu-\frac{n}{2}}(h^*) - \operatorname{op}_M^{-\gamma_+ -\frac{n}{2}}(h^*) \right\} \omega(t[\lambda]^{\frac{1}{\mu}}).$$

Here, $\gamma_+ = \gamma$ if h^* has no pole on the line $\mathrm{Re}\, z = \frac{n+1}{2} + \gamma$, otherwise $\gamma_+ > \gamma$ sufficiently close to γ. However, it is known, cf. [**5**], Section 8.1.2, Theorem 6, that then $G_1 \in R^{-\mu,\mu}_G(X^\wedge; \Lambda, (-\gamma, -\gamma))$ and

$$M^* - G_1 \in S^{-\mu,\mu}_{cl}(\Lambda; \mathcal{K}^{s,-\gamma}(X^\wedge)^\nu, \mathcal{K}^{s',-\gamma_+ +\mu}(X^\wedge)^{\nu'})$$

for all s, s', ν, ν'. All together this shows that $M \in R^{-\mu,\mu}_G(X^\wedge; \Lambda, \gamma)$ and hence justifies the description of the resolvent we have given in Theorem 4.6.

9. Notation

For $0 \neq \lambda \in \mathbb{C}$ we let $\arg\lambda$ be the unique number $-\pi \leq \arg\lambda < \pi$ such that $\lambda = |\lambda| e^{i\arg\lambda}$. For $z \in \mathbb{C}$ we then set

$$\lambda^z = |\lambda|^z e^{iz\arg\lambda}.$$

For fixed z this is a holomorphic function in $\lambda \in \mathbb{C} \setminus \{\lambda \in \mathbb{R} \mid \lambda \leq 0\}$.

For $\delta > 0$ and $0 < \theta < \pi$ we let $\Lambda = \Lambda(\delta,\theta)$ denote the closed *keyhole region*

$$\Lambda(\delta,\theta) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq \delta \text{ or } |\arg\lambda| \geq \theta\}$$

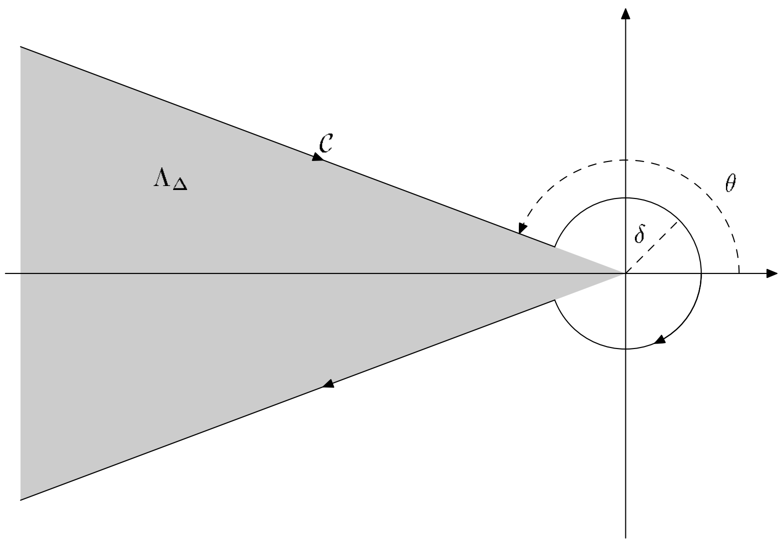
and $\mathcal{C} = \mathcal{C}(\delta,\theta)$ its parametrized boundary, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, with

$$\mathcal{C}_1(t) = te^{i\theta},\ -\infty < t \leq \delta; \quad \mathcal{C}_2(t) = \delta e^{-it},\ -\theta \leq t \leq \theta; \quad \mathcal{C}_3(t) = te^{-i\theta},\ \delta \leq t < \infty. \tag{9.1}$$

We let $\Lambda_\Delta = \Lambda_\Delta(\theta)$ denote the closed sector contained in Λ,

$$\Lambda_\Delta(\theta) = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \geq \theta\} \cup \{0\} \tag{9.2}$$

and, similar to (9.1), $\mathcal{C}_\Delta$ its parametrized boundary.



We now recall various spaces of pseudodifferential symbols and operators we shall use throughout this paper. In the following we let $\mu, d \in \mathbb{R}$ and d positive.

We call a function smooth on Λ, if it is the restriction to Λ of a function which is smooth in an open neighborhood of Λ. If E is a Fréchet space, then

$$\mathcal{S}(\Lambda, E) \tag{9.3}$$

consists of all $u \in \mathcal{C}^\infty(\Lambda, E)$ satisfying

$$\sup_{\lambda\in\Lambda} |||\partial_\lambda^\gamma u(\lambda)|||\, |\lambda|^N < \infty$$

for any multi-index $\gamma \in \mathbb{N}_0^2$, any $N \in \mathbb{N}$, and any continuous semi-norm $|||\cdot|||$ of E. The space of symbols of order μ and anisotropy d,

$$S^{\mu,d}(\mathbb{R}^m_y \times \mathbb{R}^n_\eta; \Lambda),$$

consists of all functions $a \in \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \Lambda)$, which fulfill the estimates

$$|\partial_y^\beta \partial_\eta^\alpha \partial_\lambda^\gamma a(y,\eta,\lambda)| \leq c_{\alpha\beta\gamma}\ \langle \eta, \lambda\rangle_d^{\mu-|\alpha|-d|\gamma|}, \qquad \langle\eta,\lambda\rangle_d = (1+|\eta|^2+|\lambda|^{\frac{2}{d}})^{\frac{1}{2}},$$

for any multi-indices α, β, and γ. Further we set

$$S^{\mu,d}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{m-1} \times \mathbb{R}^n;\Lambda) = S^{\mu,d}(\mathbb{R}^m \times \mathbb{R}^n;\Lambda)|_{\overline{\mathbb{R}}_+\times\mathbb{R}^{m-1}\times\mathbb{R}^n\times\Lambda}.$$

For a compact manifold X, $\dim X = n$, the space

$$L^{\mu,d}(X;\Lambda) \tag{9.4}$$

of parameter-dependent pseudodifferential operators of order μ and anisotropy d consists of all operator-families, which are obtained as a sum (according to a covering of X by coordinate neighborhoods) of local operators with symbols from $S^{\mu,d}(\mathbb{R}^n\times\mathbb{R}^n;\Lambda)$ and a smoothing remainder from $L^{-\infty}(X;\Lambda) := \mathcal{S}(\Lambda, L^{-\infty}(X))$. In the last definition, $L^{-\infty}(X)$ is the usual space of smoothing operators on X, i.e. the space of all integral operators having a smooth kernel.

If $\gamma \in \mathbb{R}$ and Γ_γ denotes the vertical line in the complex plane

$$\Gamma_\gamma = \{z \in \mathbb{C} \mid \operatorname{Re} z = \gamma\},$$

the space of symbols

$$MS^\mu(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$$

consists of all functions $a \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$ which satisfy the estimates

$$|\partial_\tau^l (t\partial_t)^k \partial_\xi^\alpha \partial_x^\beta a(t,x,\tfrac{n+1}{2}-\gamma+i\tau,\xi)| \le c_{kl\alpha\beta}\langle\tau,\xi\rangle^{\mu-l-|\alpha|}, \qquad \langle\tau,\xi\rangle = (1+\tau^2+|\xi|^2)^{1/2}.$$

The associated (Fourier-Mellin) pseudodifferential operator is

$$[\mathrm{op}_M^{\gamma-\frac{n}{2}}(a)u](t,x) = \int_{\operatorname{Re} z = \frac{n+1}{2}-\gamma} t^{-z}\,\mathrm{op}(a)(t,z)(\mathcal{M}u)(z,x)\,dz, \qquad u \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+\times\mathbb{R}^n),$$

where op is the standard Fourier pseudodifferential operator on $\mathbb{R}^n$, and $\mathcal{M}$ the Mellin transform

$$(\mathcal{M}v)(z) = \int_0^\infty t^z v(t)\,\tfrac{dt}{t}.$$

Such operators induce continuous mappings $\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)$ for Sobolev spaces

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n) := \big\{u \mid (Su)(r,x) := e^{(\frac{n}{2}-\gamma)r}u(e^{-r},x) \in H^s(\mathbb{R}^{1+n}_{(r,x)})\big\}.$$

The continuity is due to the fact that a Mellin pseudodifferential operator on $\mathbb{R}_+\times\mathbb{R}^n$ transforms, under conjugation by S, to a usual pseudodifferential operator on $\mathbb{R}^{1+n}$, and then the Calderón-Vaillancourt theorem applies.

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