The Dirac Determinant of Spherical Space Forms

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1 Introduction

In classical field theory the physical fields φ have to satisfy certain field equations or, equivalently, have to be critical points of some action functional $\mathcal{S}[\varphi]$. When passing to quantum field theory this requirement is discarted and one instead looks at partition functions defined as a functional integral

$$Z = \int \exp(-\mathcal{S}[\varphi])\mathcal{D}\varphi$$

over the space of all fields. It is now a serious problem that in most cases the space of fields is infinite dimensional and the "measure" $\mathcal{D}\varphi$ does not exist. If the space of fields is a Hilbert space and the action is of the form $\mathcal{S}[\varphi] = \frac{1}{2}(L\,\varphi, \varphi)$ where L is a positive self-adjoint operator, then one can *define* the functional integral by

$$Z = (\det L)^{-1/2}.$$

This is motivated by the fact that if the Hilbert space is of finite dimension N, then

$$\int_{\mathbb{R}^N} \exp(-\frac{1}{2}(L\,x, x))dx^1 \cdots dx^N = (2\pi)^{N/2} \det(L)^{-1/2}.$$

One can then regard $\mathcal{D}x = \frac{dx^1}{\sqrt{2\pi}} \cdots \frac{dx^N}{\sqrt{2\pi}}$ as a renormalized Lebesgue measure. At first it may seem that one has simply shifted the problem since in the physical case the operator L will typically have unbounded spectrum and the product of its eigenvalues will diverge. So one has to find a reasonable definition for the determinant of L. We will look at two closely related definitions for the determinant, the one most commonly used is coming from the ζ-function of L while the proper time regularized determinant makes essential use of the asymptotic heat kernel expansion. This will be explained in detail in the next section. In order to distinguish these regularized determinants from the usual ones in finite dimensions we will denote them by DET and by $\mathrm{DET}_{\mathrm{p.t.}}$.

Even though these concepts are standard in quantum field theory not many of these determinants have been computed explicitly. Due to the somewhat involved nature of their definition an explicit computation can be achieved only in cases of high symmetry. In this paper we provide such explicit computation for the Dirac operator and its square on spherical space forms. The square of the Dirac operator D^2 is a non-negative self-adjoint elliptic differential operator acting on spinor fields. In Theorem 4.1 we present a formula for the determinant of D^2 on the n-dimensional sphere, $n \geq 2$, with its standard Riemannian metric of constant curvature 1. The determinant is given by a linear combination of the Riemann ζ-function and its first derivative evaluated at certain non-positive integers. The proper time regularized determinant is given in Corollary 4.3.

Since the Dirac operator D itself has a spectrum unbounded from above and from below one has to find a reasonable definition for its determinant. Since regularized determinants are in general not multiplicative,

$$\mathrm{DET}(A\,B) \neq \mathrm{DET}(A)\,\mathrm{DET}(B),$$

it is not sufficient to simply set $\mathrm{DET}(D) := \sqrt{\mathrm{DET}(D^2)}$. In (3) we define $\mathrm{DET}(D)$ to be of the form $\exp(i\varphi)\sqrt{\mathrm{DET}(D^2)}$ where the ζ-invariant of D^2 and the η-invariant of D enter into the phase φ. This definition is motivated in Section 2. We give a simple expression for the *multiplicative anomaly* $\exp(i\varphi)$ in Theorem 4.2. It turns out to be trivial for odd-dimensional spheres. The case of the 1-dimensional sphere needs special treatment since S^1 has two different spin structures. The results for S^1 are contained in Theorem 4.4 and turn out to depend on the choice of spin structure. The numerical values strongly suggest that the Dirac determinant on the n-dimensional sphere tends to 1 as the dimension n goes to ∞,

$$\lim_{n\to\infty} \mathrm{DET}(D; S^n) = 1.$$

We have no rigorous proof for this conjecture.

We then pass to spherical space forms, i. e. to quotients of the sphere. We first look at the determinant of D^2 and compute the *covering anomaly*

$$\frac{\mathrm{DET}(D^2; G\backslash S^n)}{\mathrm{DET}(D^2; S^n)^{\frac{1}{|G|}}}$$

in Theorem 5.1. Here only odd dimensions n have to be considered. For the example of real-projective space it turns out that the covering anomaly is trivial,

$$\mathrm{DET}(D^2; \mathbb{RP}^n) = \sqrt{\mathrm{DET}(D^2; S^n)}.$$

In Theorem 5.2 we give the formula for the determinant of D itself on spherical space forms. For the computation of the multiplicative anomaly we need to recall a formula for the η-invariant of spherical space forms which we do in Theorem 3.2.

In the final section we restrict our attention to S^2 but we twist the Dirac operator with a complex line bundle. The Chern number of such a bundle can be interpreted as a topological charge. In Theorem 6.1 we compute the eigenvalues of these twisted Dirac operators and in Theorem 6.2 we compute their determinants. As a special case this includes the determinant of the Laplace-Beltrami operator acting on functions.

It is always understood that the spherical space forms are equipped with their standard metrics of constant curvature 1. On S^2, S^4, and S^6 the standard metric is known to have an interesting criticality property for the determinant of D^2 [4, 5, 13, 15].

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2 Determinant of the Dirac Operator

Throughout this paper let L be a non-negative self-adjoint elliptic differential operator of second order acting on sections of a Riemannian or Hermitian vector bundle over a compact n-dimensional Riemannian manifold M. See [10] and [14] for the basics on spin and spectral geometry. The most common regularization scheme for making sense of the determinant of L is the definition via its *ζ-function* which reads

$$\mathrm{DET}(L; M) := \exp\left(-\zeta_L'(0)\right) \tag{1}$$

with

$$\zeta_L(s) := \sum_{\lambda \in \mathrm{Spec}(L) - \{0\}} \frac{1}{\lambda^s}.$$

This way of defining the determinant is motivated by the observation that for positive numbers λ_k we have $\ln \lambda_k = -\frac{d}{ds}\lambda_k^{-s}|_{s=0}$ and $\exp\left(\sum_k \ln \lambda_k\right) = \prod_k \lambda_k$. The series defining $\zeta_L(s)$ converges for $\Re(s) > n/2$. It can be shown that ζ_L extends meromorphically to $\mathbb{C}$ and that it has no pole at $s = 0$.

Note that this definition of the determinant excludes all information on the possible eigenvalue zero. It has to be taken into account separately.

There is the following expression of $\zeta_L(s)$ in terms of its Mellin transform:

$$\zeta_L(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{tr}(e^{-tL} - P_0)dt \tag{2}$$

where P_0 denotes the projector onto the kernel of L.

So far we have a definition of the determinant only in case the operator L is non-negative. This can be applied to the square of the Dirac operator, $L = D^2$, but not to the Dirac operator itself. We adopt the following convention for the determinant of D [20]:

$$\mathrm{DET}(D; M) := \exp\left(i\frac{\pi}{2}(\zeta_{D^2}(0) - \eta_D(0))\right) \cdot \exp\left(-\frac{\zeta'_{D^2}(0)}{2}\right) \tag{3}$$

where the *η-function* is given by

$$\eta_D(s) := \sum_{\lambda \in \mathrm{Spec}(D)-\{0\}} \frac{\mathrm{sgn}\lambda}{|\lambda|^s}$$

for $\Re(s) >> 0$. What enters into the phase of the determinant is the *η-invariant* $\eta(M) := \eta_D(0)$. It measures the spectral asymmetry and is zero in case of a symmetric spectrum.

Why is the definition in (3) reasonable? Denote the positive eigenvalues of D by λ_k and the negative ones by $-\nu_k$. We can formally write down a ζ-function for D itself as follows:

$$\begin{aligned}
\zeta_D(s) &= \sum_k \lambda_k^{-s} + \sum_k (-1)^{-s}\nu_k^{-s} \\
&= \sum_k \left(\frac{\lambda_k^{-s} + \nu_k^{-s}}{2} + \frac{\lambda_k^{-s} - \nu_k^{-s}}{2}\right) + (-1)^{-s} \sum_k \left(\frac{\lambda_k^{-s} + \nu_k^{-s}}{2} - \frac{\lambda_k^{-s} - \nu_k^{-s}}{2}\right) \\
&= \frac{\zeta_{D^2}(\frac{s}{2}) + \eta_D(s)}{2} + (-1)^{-s}\, \frac{\zeta_{D^2}(\frac{s}{2}) - \eta_D(s)}{2}.
\end{aligned}$$

This is a meromorphic function well-defined up to the sign ambiguity in $(-1)^{-s} = e^{\mp i\pi s}$. Choosing $(-1)^{-s} = e^{-i\pi s}$ we get

$$\exp\left(-\zeta'_D(0)\right) = \exp\left(i\frac{\pi}{2}(\zeta_{D^2}(0) - \eta_D(0))\right) \cdot \exp\left(-\frac{\zeta'_{D^2}(0)}{2}\right)$$

thus yielding (3).

We want to compare the above considerations with another way of defining a regularized determinant. For $\varepsilon > 0$ define

$$\ln \mathrm{DET}_\varepsilon(L; M) := -\int_\varepsilon^\infty t^{-1}\, \mathrm{tr}(e^{-tL} - P_0)dt. \tag{4}$$

Since the integrand decays exponentially fast the integral is finite for every positive ε. One says that $\mathrm{DET}_\varepsilon(L; M)$ is obtained from the (infinite) determinant of L by cutoff in proper time [17, p. 170]. To motivate this definition note that (2) gives for $\Re(s) >> 0$

$$\zeta_L'(s) = -\frac{\Gamma'(s)}{\Gamma(s)^2}\int_0^\infty t^{s-1}\mathrm{tr}(e^{-tL}-P_0)dt + \frac{1}{\Gamma(s)}\int_0^\infty \ln(t)t^{s-1}\mathrm{tr}(e^{-tL}-P_0)dt.$$

After replacing the lower integral boundary 0 by $\varepsilon > 0$ we can take the limit $s \to 0$. The Laurent expansion

$$\Gamma(s) = \frac{1}{s} + \mathrm{O}(1) \tag{5}$$

gives $\lim_{s\to 0}\frac{1}{\Gamma(s)} = 0$ and $\lim_{s\to 0}\left(-\frac{\Gamma'(s)}{\Gamma(s)^2}\right) = \frac{d}{ds}\Gamma(s)^{-1}|_{s=0} = 1$. This way we obtain the right hand side of (4).

The proper time regularized determinant is now defined as the "finite part" of $\mathrm{DET}_\varepsilon(L;M)$ for $\varepsilon \searrow 0$. To make this more precise we look at the asymptotic expansion of the heat kernel of $L = D^2$ for $t \searrow 0$

$$\mathrm{tr}\, e^{-tL} \sim \sum_{k=0}^{\infty} \Phi_{k-\frac{n}{2}}(L)\, t^{k-\frac{n}{2}}.$$

In particular, we can plug

$$\mathrm{tr}\, e^{-tL} = \sum_{k=0}^{[n/2]} \Phi_{k-\frac{n}{2}}(L) t^{k-\frac{n}{2}} + R(t) \tag{6}$$

into (4) where the remainder term $R(t)$ is of order $\mathrm{O}(t)$ if n is even and of order $\mathrm{O}(\sqrt{t})$ if n is odd. After splitting the integral in (4) into one over $[\varepsilon, 1]$ and one over $[1,\infty]$ we get

$$\begin{aligned}\ln \mathrm{DET}_\varepsilon(L;M) &= -\int_\varepsilon^1 \left(\sum_{k=0}^{[n/2]} \Phi_{k-\frac{n}{2}}(L)t^{k-\frac{n}{2}-1} + R(t)t^{-1} - \dim\ker(L)t^{-1}\right)dt \\ &\quad -\int_1^\infty t^{-1}\mathrm{tr}(e^{-tL}-P_0)dt \\ &= -\sum_{k=0}^{[(n-1)/2]} \frac{\Phi_{k-\frac{n}{2}}(L)}{k-\frac{n}{2}} + \sum_{k=0}^{[(n-1)/2]} \frac{\Phi_{k-\frac{n}{2}}(L)\varepsilon^{k-\frac{n}{2}}}{k-\frac{n}{2}} + (\Phi_0(L) \\ &\quad - \dim\ker(L))\ln\varepsilon - \int_\varepsilon^1 R(t)t^{-1}dt - \int_1^\infty t^{-1}\mathrm{tr}(e^{-tL}-P_0)dt.\end{aligned}$$

Here we use the convention $\Phi_0(L) = 0$ if n is odd. The terms $\frac{\Phi_{k-\frac{n}{2}}(L)}{k-\frac{n}{2}}\varepsilon^{k-\frac{n}{2}}$, $k \le [(n-1)/2]$, and $(\Phi_0(L) - \dim\ker(L))\ln\varepsilon$ explode for $\varepsilon \searrow 0$ unless they vanish. We now abandon the terms divergent in the limit $\varepsilon \searrow 0$ and we are led to

Definition 1. The *proper time regularized determinant* $\mathrm{DET}_{\mathrm{p.t.}}(L;M)$ is defined by

$$\begin{aligned}\ln \mathrm{DET}_{\mathrm{p.t.}}(L;M) := \sum_{k=0}^{[(n-1)/2]} \frac{\Phi_{k-\frac{n}{2}}(L)}{\frac{n}{2}-k} - \int_0^1 R(t)t^{-1}dt \\ - \int_1^\infty t^{-1}\mathrm{tr}(e^{-tL}-P_0)dt\end{aligned} \tag{7}$$

where $\Phi_{k-\frac{n}{2}}(L)$ and $R(t)$ are as in (6).

The following relation (cf. [17, (28.11), p. 171]) shows that the two regularizations do not differ significantly.

Proposition 2.1 *The ζ-regularized determinant $\mathrm{DET}(L;M)$ and the proper time regularized determinant $\mathrm{DET}_{\mathrm{p.t.}}(L;M)$ are related by*

$$\ln \mathrm{DET}(L;M) - \ln \mathrm{DET}_{\mathrm{p.t.}}(L;M) = \Gamma'(1)(\Phi_0(L) - \dim\ker L) = \Gamma'(1)\zeta_L(0).$$

Proof. In order to compare (7) with the ζ-determinant we insert the heat kernel expansion (6) into (2) and do the corresponding integral splitting into one over $[0,1]$ and one over $[1,\infty]$.

$$\zeta_L(s) = \frac{1}{\Gamma(s)}\left(\sum_{k=0}^{[(n-1)/2]} \frac{\Phi_{k-\frac{n}{2}}(L)}{s-k+\frac{n}{2}} + \frac{\Phi_0(L) - \dim\ker L}{s} + \int_0^1 R(t)t^{s-1}dt \right.$$
$$\left. + \int_1^\infty t^{s-1}\mathrm{tr}(e^{-tL} - P_0)dt\right). \tag{8}$$

Using (5) we find

$$\zeta_L(0) = \Phi_0(L) - \dim\ker L.$$

Equation (8) gives us for the determinant

$$\ln \mathrm{DET}(L;M) = -\zeta_L'(0) = \sum_{k=0}^{[(n-1)/2]} \frac{\Phi_{k-\frac{n}{2}}(L)}{\frac{n}{2}-k} + \Gamma'(1)\zeta_L(0) - \int_0^1 R(t)t^{-1}dt$$
$$- \int_1^\infty t^{-1}\mathrm{tr}(e^{-tL} - P_0)dt \tag{9}$$

Comparing (7) and (9) we get Proposition 2.1.

3 Dirac Spectrum of Spherical Space Forms

The Dirac spectrum of the sphere S^n, $n \geq 2$, with constant curvature 1 has been computed by different methods in [2, 7, 18, 19]. The eigenvalues are

$$\pm\left(\frac{n}{2} + k\right), \tag{10}$$

$k \in \mathbb{N}_0$, with multiplicity $2^{[n/2]}\begin{pmatrix} k+n-1 \\ k \end{pmatrix}$. For $n \geq 2$ the sphere is simply connected, hence has only one spin structure. It is given by the standard projection $\mathrm{Spin}(n+1) \to \mathrm{Spin}(n+1)/\mathrm{Spin}(n) = S^n$.

We now look at spherical space forms $M = G\backslash S^n$ where G is a finite fixed point free subgroup of $\mathrm{SO}(n{+}1)$. Spin structures correspond to homomorphisms $\varepsilon : G \to \mathrm{Spin}(n{+}1)$ such that the diagram

$$\begin{tikzcd} & \mathrm{Spin}(n+1) \arrow[d] \\ G \arrow[ur, "\varepsilon"] \arrow[r] & \mathrm{SO}(n+1) \end{tikzcd}$$

commutes. The corresponding spin structure is given by $\varepsilon(G)\backslash\mathrm{Spin}(n{+}1) \to G\backslash(\mathrm{Spin}(n{+}1)/\mathrm{Spin}(n)) = M$. Thus spinors on M correspond to $\varepsilon(G)$-invariant spinors on S^n.

Since any eigenspinor on M can be lifted to S^n all eigenvalues of M are also eigenvalues of S^n, hence of the form (10). To know the spectrum of M one must compute the multiplicities μ_k of $\frac{n}{2}+k$ and μ_{-k} of $-(\frac{n}{2}+k)$. It is convenient to encode them into two power series, so-called *Poincaré series*

$$F_+(z) := \sum_{k=0}^{\infty} \mu_k z^k,$$
$$F_-(z) := \sum_{k=0}^{\infty} \mu_{-k} z^k.$$

To formulate the result recall that in even dimension $2m$ the spinor representation is reducible and can be decomposed into two half spinor representations

$$\mathrm{Spin}(2m) \to \mathrm{Aut}(\Sigma_{2m}^{\pm}),$$

$\Sigma_{2m} = \Sigma_{2m}^+ \oplus \Sigma_{2m}^-$. Denote their characters by $\chi^{\pm} : \mathrm{Spin}(2m) \to \mathbb{C}$.

Theorem 3.1 ([2]) *Let $M = G\backslash S^n$, $n = 2m-1$, be a spherical space form with spin structure given by $\varepsilon : G \to \mathrm{Spin}(2m)$. Then the eigenvalues of the Dirac operator are $\pm(\frac{n}{2}+k)$, $k \geq 0$, with multiplicities determined by*

$$F_+(z) = \frac{1}{|G|}\sum_{g\in G} \frac{\chi^-(\varepsilon(g)) - z\cdot\chi^+(\varepsilon(g))}{\det(1_{2m} - z\cdot g)},$$
$$F_-(z) = \frac{1}{|G|}\sum_{g\in G} \frac{\chi^+(\varepsilon(g)) - z\cdot\chi^-(\varepsilon(g))}{\det(1_{2m} - z\cdot g)}.$$

Note that only odd-dimensional spherical space forms are of interest because in even dimensions real projective space is the only quotient and in this case it is not even orientable.

This kind of encoding the spectrum of M is in fact well-suited for computation of the η-invariant.

Theorem 3.2 ([3]) *Let $M = G\backslash S^{2m-1}$ be a spherical space form with spin structure given by $\varepsilon : G \to \mathrm{Spin}(2m)$. Then the η-invariant of M is given by*

$$\eta(G\backslash S^{2m-1}) = \frac{2}{|G|}\sum_{g\in G-\{1_{2m}\}} \frac{(\chi^- - \chi^+)(\varepsilon(g))}{\det(1_{2m} - g)}.$$

For the convenience of the reader and since we will need the arguments again we briefly sketch the proof. We define the *θ-functions*

$$\begin{aligned}\theta_\pm(t) &:= e^{-\frac{n}{2}t}\cdot F_\pm(e^{-t})\\ &= \frac{e^{-(m+\frac{1}{2})t}}{|G|}\sum_{g\in G}\frac{\chi^{\mp}(\varepsilon(g)) - e^{-t}\cdot\chi^{\pm}(\varepsilon(g))}{\det(1_{2m} - e^{-t}\cdot g)}.\end{aligned} \tag{11}$$

Since G acts freely the non-trivial elements $g \in G$ do not have 1 as an eigenvalue and thus $\det(1_{2m} - g) \neq 0$. Therefore only the summand for $g = 1_{2m}$ contributes to the pole of $\theta_\pm$ at $t = 0$. Hence for $\theta := \theta_+ - \theta_-$ the poles at $t = 0$ cancel, θ is holomorphic at $t = 0$ with

$$\theta(0) = \frac{2}{|G|}\sum_{g\in G-\{1_{2m}\}} \frac{(\chi^- - \chi^+)(\varepsilon(g))}{\det(1_{2m} - g)}. \tag{12}$$

Next we observe that

$$\theta_+(t) = \sum_{k=0}^{\infty} \mu_k e^{-(n/2+k)t} = \sum_{\lambda>0} e^{-\lambda t},$$

where the last sum is taken over all positive eigenvalues. Similarly for θ_-,

$$\theta_-(t) = \sum_{k=0}^{\infty} \mu_{-k} e^{-(n/2+k)t} = \sum_{\lambda<0} e^{\lambda t}.$$

Application of the Mellin transformation yields

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \theta(t) t^{s-1} dt.$$

Therefore

$$\eta(G\backslash S^{2m-1}) = \lim_{s\to 0} \frac{1}{\Gamma(s)} \int_0^\infty \theta(t) t^{s-1} dt = \mathrm{Res}_{s=0} \left(\int_0^\infty \theta(t) t^{s-1} dt \right).$$

Since θ decays exponentially fast for $t \to \infty$ the function $s \mapsto \int_1^\infty \theta(t) t^{s-1} dt$ is holomorphic at $s = 0$. Thus

$$\eta(G\backslash S^{2m-1}) = \mathrm{Res}_{s=0} \left(\int_0^1 \theta(t) t^{s-1} dt \right) = \theta(0)$$

which together with (12) proves Theorem 3.2.

Compare this to Goette's computation of the *equivariant* η-invariant of spheres in [11, Satz 6.10]. See also [9, 10] where the η-invariant of all twisted signature operators on spherical space forms is determined and used to compute their K-theory. In [8] the η-invariant of all twisted Dirac operators on 3-dimensional spherical space forms is computed.

Using Theorem 3.2 it is easy to discuss real projective space.

Corollary 3.3 ([3]) *For $n \geq 2$ real projective space $\mathbb{RP}^n$ is spin if and only if $n \equiv 3$ mod 4, in which case it has exactly two spin structures. The η-invariant for the Dirac operator is given by*

$$\eta(\mathbb{RP}^n) = \pm 2^{-m}, \quad n = 2m-1,$$

where the sign depends on the spin structure chosen.

Let us now look at lens spaces. For $\alpha \in \mathbb{R}$ define the rotation matrix

$$R(\alpha) := \begin{pmatrix} \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}.$$

For natural numbers $q, p_1, \ldots, p_m$ where p_j and q are coprime for all j, let G be the cyclic subgroup of $\mathrm{SO}(2m)$ generated by

$$\begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_m/q) \end{pmatrix}.$$

We denote the resulting lens space by $L(q, p_1, \ldots, p_m) := G\backslash S^{2m-1}$. Now Theorem 3.1 takes the form (compare [1, Satz 5.3]):

a) If q is odd, then $L(q,p_1,\dots,p_m)$ has exactly one spin structure and the Poincaré series are given by

$$F_+(z) = \frac{1}{q}\sum_{k=0}^{q-1} \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)},$$

$$F_-(z) = \frac{1}{q}\sum_{k=0}^{q-1} \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)}.$$

b) If q is even and $p_1+\ldots+p_m$ is odd (i. e. if m is odd), then $L(q,p_1,\dots,p_m)$ has no spin structures.

c) If q is even and $p_1+\ldots+p_m$ is even (i. e. if m is even), then $L(q,p_1,\dots,p_m)$ has two different spin structures. The Poincaré series for the first spin structure are given by

$$F_+(z) = \frac{1}{q}\sum_{k=0}^{q-1} \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)},$$

$$F_-(z) = \frac{1}{q}\sum_{k=0}^{q-1} \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)},$$

while for the second spin structure they are given by

$$F_+(z) = \frac{1}{q}\sum_{k=0}^{q-1}(-1)^k \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)},$$

$$F_-(z) = \frac{1}{q}\sum_{k=0}^{q-1}(-1)^k \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q) - z\cdot \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-z)(\exp(-2\pi ikp_j/q)-z)}.$$

Here the notation $\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}}$ indicates that the sum is taken over all choices of $\varepsilon_j=\pm 1$ for which $\varepsilon_1\cdots\varepsilon_m = (-1)^{m+1}$ and similarly for $(-1)^m$. For the η-invariant this yields in case a)

$$\eta = \theta(0) = \frac{2}{q}\sum_{k=1}^{q-1} \frac{\sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m+1}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q) - \sum\limits_{\substack{\varepsilon_1\cdots\varepsilon_m\\=(-1)^{m}}} \exp(\pi ik(q+1)\sum_j \varepsilon_j p_j/q)}{\prod_{j=1}^m (\exp(2\pi ikp_j/q)-1)(\exp(-2\pi ikp_j/q)-1)}$$

$$= (-1)^{m+1}\,\frac{2}{q}\sum_{k=1}^{q-1}\frac{\sum\limits_{\varepsilon_1,\dots,\varepsilon_m}\prod\limits_{j=1}^{m}\varepsilon_j\exp(\pi ik(q+1)\varepsilon_j p_j/q)}{\prod\limits_{j=1}^{m}(2-2\cos(2\pi kp_j/q))}$$

$$= (-1)^{m+1}\,\frac{2}{q}\sum_{k=1}^{q-1}\frac{\prod\limits_{j=1}^{m}(\exp(\pi ik(q+1)p_j/q)-\exp(-\pi ik(q+1)p_j/q))}{2^m\prod\limits_{j=1}^{m}(1-\cos(2\pi kp_j/q))}$$

$$= (-1)^{m+1}\,i^m\,\frac{2}{q}\sum_{k=1}^{q-1}\prod_{j=1}^{m}\frac{\sin(\pi k(q+1)p_j/q)}{1-\cos(2\pi kp_j/q)}.$$

If m is odd we see that η is imaginary, but on the other hand the η-invariant is always real. Hence it must vanish in this case. In fact, the η-invariant of the Dirac operator always vanishes unless the dimension of the manifold is $n \equiv 3 \bmod 4$. Case c) is treated similarly. Case b) cannot occur for even m. We summarize:

Corollary 3.4 *Let $m \in \mathbb{N}$ be even and let $q, p_1, \dots, p_m \in \mathbb{N}$ be such that q and p_j are coprime for all j.*

Then if q is odd $L(q, p_1, \dots, p_m)$ has exactly one spin structure and the η-invariant is given by

$$\eta(L(q,p_1,\dots,p_m)) = (-1)^{3m/2+1}\,\frac{2}{q}\sum_{k=1}^{q-1}\prod_{j=1}^{m}\frac{\sin(\pi k(q+1)p_j/q)}{1-\cos(2\pi kp_j/q)}.$$

If q is even, then $L(q, p_1, \dots, p_m)$ has two different spin structures. For the first spin structure the η-invariant is given by

$$\eta(L(q,p_1,\dots,p_m)) = (-1)^{3m/2+1}\,\frac{2}{q}\sum_{k=1}^{q-1}\prod_{j=1}^{m}\frac{\sin(\pi kp_j/q)}{1-\cos(2\pi kp_j/q)}$$

while for the second it is

$$\eta(L(q,p_1,\dots,p_m)) = (-1)^{3m/2+1}\,\frac{2}{q}\sum_{k=1}^{q-1}(-1)^k\prod_{j=1}^{m}\frac{\sin(\pi kp_j/q)}{1-\cos(2\pi kp_j/q)}.$$

The case $q = 2$ and $p_1 = \dots = p_m = 1$ recovers Corollary 3.3. We collect a few examples in Tables 1 and 2. For even q we denote the η-invariants for the two spin structures by η_1 and η_2.

4 Dirac Determinant of the Sphere

In this section we compute the determinant of the square of the Dirac operator and of the Dirac operator itself over the sphere of constant sectional curvature 1. For this task one could use the general machinery developped in [6] for homogeneous operators on compact locally symmetric spaces. We found a direct approach more convenient. Recall the *Riemann ζ-function* ζ_R and the *Hurwitz ζ-function* ζ_H defined by meromorphic extension of the series

$$\zeta_R(s) = \sum_{j=1}^{\infty}\frac{1}{j^s} \qquad \text{and} \qquad \zeta_H(s,b) = \sum_{j=0}^{\infty}\frac{1}{(j+b)^s}$$

Table 1.

M	$\dim(M)$	η
$L(3,1,1)$		$\frac{4}{9}$
$L(3,1,2)$		$-\frac{4}{9}$
$L(5,1,1)$	3	$\frac{4}{5}$
$L(5,1,2)$		0
$L(5,2,3)$		$-\frac{4}{5}$
$L(3,1,1,1,1)$		$-\frac{4}{27}$
$L(3,1,1,1,2)$	7	$\frac{4}{27}$
$L(5,1,1,1,1)$		$-\frac{12}{25}$
$L(3,1,1,1,1,1,1)$		$\frac{4}{81}$
$L(3,1,1,1,1,1,2)$	11	$-\frac{4}{81}$
$L(5,1,1,1,1,1,1)$		$\frac{8}{25}$
$L(3,1,1,1,1,1,1,1,1)$		$-\frac{4}{243}$
$L(3,1,1,1,1,1,1,1,2)$	15	$\frac{4}{243}$
$L(5,1,1,1,1,1,1,1,1)$		$-\frac{28}{125}$
$L(3,1,1,1,1,1,1,1,1,1,1)$		$\frac{4}{729}$
$L(3,1,1,1,1,1,1,1,1,1,2)$	19	$-\frac{4}{729}$
$L(5,1,1,1,1,1,1,1,1,1,1)$		$\frac{4}{25}$

Table 2.

M	$\dim(M)$	η_1	η_2
$L(4,1,1)$		$\frac{5}{8}$	$-\frac{3}{8}$
$L(4,1,3)$		$\frac{3}{8}$	$-\frac{5}{8}$
$L(6,1,1)$		$\frac{35}{36}$	$-\frac{19}{36}$
$L(6,1,5)$		$\frac{19}{36}$	$-\frac{35}{36}$
$L(8,1,1)$	3	$\frac{21}{16}$	$-\frac{11}{16}$
$L(8,1,3)$		$\frac{3}{16}$	$\frac{3}{16}$
$L(8,1,5)$		$-\frac{3}{16}$	$-\frac{3}{16}$
$L(8,3,5)$		$\frac{11}{16}$	$-\frac{21}{16}$
$L(10,1,1)$		$\frac{33}{20}$	$-\frac{17}{20}$
$L(4,1,1,1,1)$		$-\frac{9}{32}$	$\frac{7}{32}$
$L(4,1,1,1,3)$		$-\frac{7}{32}$	$\frac{9}{32}$
$L(6,1,1,1,1)$	7	$-\frac{329}{432}$	$\frac{265}{432}$
$L(8,1,1,1,1)$		$-\frac{105}{64}$	$\frac{87}{64}$
$L(10,1,1,1,1)$		$-\frac{1221}{400}$	$\frac{1029}{400}$
$L(4,1,1,1,1,1,1)$		$\frac{17}{128}$	$-\frac{15}{128}$
$L(6,1,1,1,1,1,1)$	11	$\frac{3611}{5184}$	$-\frac{3355}{5184}$
$L(8,1,1,1,1,1,1)$		$\frac{657}{256}$	$-\frac{623}{256}$

with $b > 0$. Observe that the binomial coefficient $M_n(x) := \binom{x+n-1}{x} = \frac{(x+n-1)\cdots(x+1)}{(n-1)!}$ is a polynomial in x of degree $n-1$. Hence we can define

$$A(k,n) := \frac{1}{k!}\frac{d^k}{dx^k}M_n(x)|_{x=-\frac{n}{2}} \tag{13}$$

or, equivalently,

$$M_n(x) = \sum_{k=0}^{n-1} A(k,n)\left(x+\frac{n}{2}\right)^k. \tag{14}$$

Moreover, we set

$$C(k,n) := \begin{cases} \sum_{j=1}^{m-1} j^k \ln(j), & \text{if } n = 2m \text{ is even,} \\ \sum_{j=0}^{m-1}(j+\frac{1}{2})^k \ln(j+\frac{1}{2}), & \text{if } n = 2m+1 \text{ is odd.} \end{cases} \tag{15}$$

Now we can formulate the result for D^2 which in different notation is also contained in [4, Section 8].

Theorem 4.1 *The determinant of the square of the Dirac operator on the n-dimensional sphere S^n, $n \geq 2$, with constant sectional curvature 1 is given by the following formulas:*

If $n = 2m$ is even, then

$$\ln \mathrm{DET}(D^2; S^{2m}) = -2^{m+2}\sum_{k=0}^{n-1} A(k,n)\,(\zeta_R'(-k) + C(k,n)), \tag{16}$$

and if $n = 2m+1$ is odd, then

$$\ln \mathrm{DET}(D^2; S^{2m+1}) = -2^{m+2}\sum_{k=0}^{n-1} A(k,n)\left(\zeta_R'(-k)(\frac{1}{2^k}-1) + \frac{\ln 2}{2^k}\,\zeta_R(-k) + C(k,n)\right) \tag{17}$$

where ζ_R is the Riemann ζ-function, $A(k,n)$ is as in (13) and $C(k,n)$ is as in (15).

Proof. From (10) we see that on S^n

$$\zeta_{D^2}(s) = 2^{[n/2]+1}\sum_{j=0}^{\infty}\frac{\binom{j+n-1}{j}}{(j+\frac{n}{2})^{2s}}.$$

By (14) we obtain

$$\begin{aligned}\zeta_{D^2}(s) &= 2^{[n/2]+1}\sum_{k=0}^{n-1} A(k,n)\sum_{j=0}^{\infty}(j+\frac{n}{2})^{k-2s} \\ &= 2^{[n/2]+1}\sum_{k=0}^{n-1} A(k,n)\,\zeta_H(2s-k,\frac{n}{2}).\end{aligned} \tag{18}$$

In case $n = 2m$ is even we use

$$\zeta_H(s,m) = \zeta_R(s) - \sum_{j=1}^{m-1} j^{-s}$$

valid for $m \in \mathbb{N}$ to get

$$\zeta'_{D^2}(0) = 2^{m+2} \sum_{k=0}^{n-1} A(k,n) \left(\zeta'_R(-k) + \sum_{j=1}^{m-1} j^k \ln(j) \right).$$

This proves (16). For $n = 2m+1$ odd we obtain

$$\zeta'_{D^2}(0) = 2^{m+2} \sum_{k=0}^{n-1} A(k,n) \zeta'_H(-k, m+\frac{1}{2}).$$

From

$$\zeta_H(s, m+\frac{1}{2}) = \zeta_H(s, \frac{1}{2}) - \sum_{j=0}^{m-1} (j+\frac{1}{2})^{-s},$$

valid for $m \in \mathbb{N}$ and

$$\zeta_H(s, \frac{1}{2}) = (2^s - 1)\zeta_R(s)$$

(see e.g. [12, 9.535]) we get (17).

Example 1. From this theorem we get

$$\begin{aligned}
\mathrm{DET}(D^2; S^2) &= \exp(-8\,\zeta'_R(-1)),\\
\mathrm{DET}(D^2; S^3) &= 2^{-\frac{1}{2}} \exp\left(3\,\zeta'_R(-2)\right)),\\
\mathrm{DET}(D^2; S^4) &= \exp\left(\frac{8}{3}\zeta'_R(-1) - \frac{8}{3}\zeta'_R(-3)\right),\\
\mathrm{DET}(D^2; S^5) &= 2^{\frac{3}{16}} \exp\left(-\frac{5}{4}\,\zeta'_R(-2) + \frac{5}{8}\,\zeta'_R(-4)\right),\\
\mathrm{DET}(D^2; S^6) &= \exp\left(-\frac{16}{15}\,\zeta'_R(-1) + \frac{4}{3}\,\zeta'_R(-3) - \frac{4}{15}\,\zeta'_R(-5)\right),\\
\mathrm{DET}(D^2; S^7) &= 2^{-\frac{5}{64}} \exp\left(\frac{259}{480}\,\zeta'_R(-2) - \frac{35}{96}\,\zeta'_R(-4) + \frac{7}{160}\,\zeta'_R(-6)\right),\\
\mathrm{DET}(D^2; S^8) &= \exp\left(\frac{16}{35}\,\zeta'_R(-1) - \frac{28}{45}\,\zeta'_R(-3) + \frac{8}{45}\,\zeta'_R(-5) - \frac{4}{315}\,\zeta'_R(-7)\right),\\
\mathrm{DET}(D^2; S^9) &= 2^{\frac{35}{1024}} \exp\left(-\frac{3229}{13440}\,\zeta'_R(-2) + \frac{47}{256}\,\zeta'_R(-4) - \frac{21}{640}\,\zeta'_R(-6)\right.\\
&\qquad \left. + \frac{17}{10752}\,\zeta'_R(-8)\right),\\
\mathrm{DET}(D^2; S^{10}) &= \exp\left(-\frac{64}{315}\,\zeta'_R(-1) + \frac{164}{567}\,\zeta'_R(-3) - \frac{13}{135}\,\zeta'_R(-5) + \frac{2}{189}\,\zeta'_R(-7)\right.\\
&\qquad \left. - \frac{1}{2835}\,\zeta'_R(-9)\right).
\end{aligned}$$

For $n = 2, 4, 6$ this reproduces the values computed by Branson in [4, Thm. 8.1]. For the Dirac operator itself we obtain

Theorem 4.2 *The determinant of the Dirac operator on the n-dimensional sphere S^n, $n \geq 2$, with constant sectional curvature 1 is given by the following:*

If $n = 2m$ is even, then

$$\mathrm{DET}(D; S^{2m}) = \exp\left(-i\pi 2^m \sum_{k=0}^{2m-1} A(k, 2m) \frac{B_{k+1}(m)}{k+1}\right) \sqrt{\mathrm{DET}(D^2; S^{2m})},$$

and if n is odd, then

$$\mathrm{DET}(D;S^n) = \sqrt{\mathrm{DET}(D^2;S^n)}$$

where $\mathrm{DET}(D^2;S^n)$ is as in Theorem 4.1, $A(k,n)$ as in (13) and $B_k(x)$ are the Bernoulli polynomials.

Proof. By (3) we have

$$\mathrm{DET}(D;S^n) = \exp\left(i\frac{\pi}{2}(\zeta_{D^2}(0) - \eta_D(0))\right)\sqrt{\mathrm{DET}(D^2;S^n)}.$$

The η-invariant vanishes because the Dirac spectrum of S^n is symmetric. It remains to compute $\zeta_{D^2}(0)$. Plugging

$$\zeta_H(-k,b) = -\frac{B_{k+1}(b)}{k+1}$$

(see [12, 9.531 and 9.623(3)]) into (18) we get

$$\zeta_{D^2}(0) = -2^{[n/2]+1}\sum_{k=0}^{n-1} A(k,n)\frac{B_{k+1}(\frac{n}{2})}{k+1}. \tag{19}$$

This proves the even-dimensional case. If n is odd we observe that $\Phi_0(D^2) = 0$ and $\ker D^2 = \{0\}$ and hence

$$\zeta_{D^2}(0) = 0. \tag{20}$$

This concludes the proof.

Using Proposition 2.1 we can combine Theorem 4.1, (19) and (20) to obtain the proper time regularized determinant of D^2 on S^n.

Corollary 4.3 *The proper time regularized determinant of the square of the Dirac operator on the n-dimensional sphere S^n, $n \geq 2$, with constant sectional curvature 1 is given by the following formulas:*

If $n = 2m$ is even, then

$$\ln \mathrm{DET}_{\mathrm{p.t.}}(D^2;S^{2m}) = -2^{m+1}\sum_{k=0}^{2m-1} A(k,2m)\left(2\,\zeta_R'(-k) + 2\,C(k,2m) - \Gamma'(1)\frac{B_{k+1}(m)}{k+1}\right)$$

and if n is odd, then

$$\mathrm{DET}_{\mathrm{p.t.}}(D^2;S^n) = \mathrm{DET}(D^2;S^n)$$

where ζ_R is the Riemann ζ-function, $A(k,n)$ is as in (13), $C(k,n)$ is as in (15) and $B_k(x)$ are the Bernoulli polynomials.

The 1-dimensional case needs to be treated separately because S^1 has two spin structures. Let us call the spin structure which extends to the unique spin structure of the 2-dimensional disk the *bounding* spin structure, the other one the *non-bounding* spin structure. For the first spin structure formula (10) still holds while the non-bounding spin structure has $\mathbb{Z}$ as its Dirac spectrum. We get

Theorem 4.4 *The determinants of the Dirac operator and its square on the 1-dimensional sphere S^1 with length 2π take the values*

$$\mathrm{DET}(D;S^1) = 2 \quad \textit{and} \quad \mathrm{DET}(D^2;S^1) = 4$$

for the bounding spin structure and

$$\mathrm{DET}(D;S^1) = -2\pi i \quad \textit{and} \quad \mathrm{DET}(D^2;S^1) = 4\pi^2$$

for the non-bounding spin structure.

Recall that by our convention for the determinant the eigenvalue 0 is ignored. Otherwise the determinant on S^1 would vanish for the non-bounding spin structure.

Proof. From (10) we get for the bounding spin structure

$$\zeta_{D^2}(s) = 2\sum_{j=0}^{\infty} \frac{1}{(j+\frac{1}{2})^{2s}} = 2\,\zeta_H(2s, \frac{1}{2}) = 2(2^{2s}-1)\zeta_R(2s).$$

Using $\zeta_R(0) = -\frac{1}{2}$ we obtain

$$\zeta'_{D^2}(0) = 4\ln 2 \cdot \zeta_R(0) = -2\ln 2$$

and hence

$$\mathrm{DET}(D^2; S^1) = \exp\left(-\zeta'_{D^2}(0)\right) = 4.$$

The computation for the non-bounding spin structure is even easier. For ζ_{D^2} we get

$$\zeta_{D^2}(s) = 2\sum_{j=1}^{\infty} \frac{1}{j^{2s}} = 2\,\zeta_R(2s).$$

Now $\zeta'_R(0) = -\frac{1}{2}\ln(2\pi)$ yields

$$\zeta'_{D^2}(0) = 4\zeta'_R(0) = -2\ln(2\pi)$$

and therefore

$$\mathrm{DET}(D^2; S^1) = 4\pi^2.$$

This proves the formulas for $\mathrm{DET}(D^2; S^1)$. The η-invariant vanishes in both cases due to spectral symmetry. The result for $\mathrm{DET}(D; S^1)$ follows from (3) and

$$\zeta_{D^2}(0) = \begin{cases} 0 & \text{for the bounding spin structure} \\ -1 & \text{for the non-bounding spin structure.} \end{cases}$$

In the following tables we give some results obtained by numerical approximation. The phase φ is defined by

$$\mathrm{DET}(D; S^n) = |\mathrm{DET}(D; S^n)| e^{i\varphi}.$$

Here are the values of the determinant on the first few even dimensional spheres.

We do not need the $\mathrm{DET}_{\mathrm{p.t.}}(D^2; S^n)$–column in the odd dimensional case because of Corollary 4.3. The phase vanishes according to Theorem 4.2.

A view at the numerical values leads us to

Conjecture 1. The determinant of the Dirac operator on the n-dimensional sphere S^n tends to 1 for $n \to \infty$,

$$\lim_{n\to\infty} \mathrm{DET}(D; S^n) = 1.$$

For example, we have

$$|\mathrm{DET}(D; S^{200})| \approx 0.99999999999999999999999999999999023251476,$$

and for the phase of $\mathrm{DET}(D; S^{200})$ we have

$$\varphi \approx 0.88944347908781047952551018696458465891 76 \cdot 10^{-31}.$$

We have no explanation for this phenomenon.

Table 3.

n	$\lvert\mathrm{DET}(D;S^n)\rvert$	φ	$\mathrm{DET}_{\mathrm{p.t.}}(D^2;S^n)$
2	1.938054383626833	-0.523598775598299	3.098641928647827
4	0.796336885184459	0.191986217719376	0.680506885334576
6	1.096240639130369	-0.0793709255073612	1.167199468652179
8	0.961240935820022	0.0345879931923003	0.935802870851431
10	1.017761616865532	-0.0155276797389130	1.029945138033507
12	0.992018956602283	0.00710558492613086	0.986674523357944
14	1.003708484448039	-0.00329528874347978	1.006211552565484
16	0.998272959376817	0.00154335558060539	0.997114236587699
18	1.000814152574965	-0.000728325704396029	1.001360932301507
20	0.999614434504206	0.000345772030034635	0.999355987552311

Table 4.

n	$\mathrm{DET}(D)$
3	0.803354268824629
5	1.090359845142337
7	0.963796369884191
9	1.016473922384390
11	0.992614518464762
13	1.003422630166412
15	0.998408322304586
17	1.000749343263366
19	0.999645452552308
21	1.000168795852563

5 Dirac Determinant of Spherical Space Forms

Next we determine the Dirac determinant of spherical space forms $M = G\backslash S^{2m-1}$ where $G \subset \mathrm{SO}(2m)$ is a fixed point free subgroup. The spin structure is given by a homomorphism $\varepsilon : G \to \mathrm{Spin}(2m)$ lifting the inclusion of G into $\mathrm{SO}(2m)$ as explained in Section 3. In order to distinguish geometric data for different manifolds we write $\zeta_{D^2}(s; M)$ for the ζ-function of the square of the Dirac operator on M.

Theorem 5.1 *Let $M = G\backslash S^{2m-1}$ be a spherical space form with spin structure given by $\varepsilon : G \to \mathrm{Spin}(2m)$. Then the square of the Dirac operator on M has the determinant*

$$\begin{aligned}\mathrm{DET}(D^2; G\backslash S^{2m-1}) \quad &= \quad \mathrm{DET}\left(D^2; S^{2m-1}\right)^{\frac{1}{|G|}} \cdot \\ \cdot \prod_{g\in G-\{1_{2m}\}} \exp &\left(-\frac{2\chi(\varepsilon(g))}{|G|} \cdot \int_0^\infty \frac{e^{-(m-\frac{1}{2})t}}{\det(1_{2m} - e^{-t}g)} \frac{1-e^{-t}}{t}\, dt\right)\end{aligned}$$

where $\mathrm{DET}(D^2; S^{2m-1})$ is given by Theorem 4.1 and χ is the character of the spinor module for $\mathrm{Spin}(2m)$.

Proof. To compute the ζ-function $\zeta_{D^2}(s\,;G\backslash S^{2m-1})$ we recall the θ-functions

$$\theta_\pm(t) \;=\; e^{-(m-\frac{1}{2})t}\cdot F_\pm(e^{-t}) \;=\; \frac{e^{-(m-\frac{1}{2})t}}{|G|}\sum_{g\in G}\frac{\chi^\mp(\varepsilon(g)) - e^{-t}\cdot\chi^\pm(\varepsilon(g))}{\det(1_{2m}-e^{-t}\cdot g)},$$

see (11). Hence by the following Mellin transformation

$$\begin{aligned}
\Gamma(2s)\,\zeta_{D^2}(s;G\backslash S^{2m-1}) &= \sum_{\lambda\in\mathrm{Spec}(D)}|\lambda|^{-2s}\int_0^\infty u^{2s-1}\,e^{-u}du\\
&= \sum_{\lambda\in\mathrm{Spec}(D)}\int_0^\infty t^{2s-1}\,e^{-|\lambda|t}dt\\
&= \sum_{k=0}^\infty(\mu_k+\mu_{-k})\int_0^\infty t^{2s-1}\,e^{-(m-\frac{1}{2}+k)t}dt\\
&= \int_0^\infty t^{2s-1}\,(\theta_+(t)+\theta_-(t))\,dt\\
&= \frac{1}{|G|}\sum_{g\in G}\chi(\varepsilon(g))\int_0^\infty\frac{t^{2s-1}\cdot e^{-(m-\frac{1}{2})t}\cdot(1-e^{-t})}{\det(1_{2m}-e^{-t}\cdot g)}dt
\end{aligned}$$

we obtain

$$\zeta_{D^2}(s;G\backslash S^{2m-1}) \;=\; \frac{1}{|G|\cdot\Gamma(2s)}\sum_{g\in G}\chi(\varepsilon(g))\int_0^\infty t^{2s}\,\frac{e^{-(m-\frac{1}{2})t}}{\det(1_{2m}-e^{-t}g)}\,\frac{1-e^{-t}}{t}dt.$$

The special case $G=\{1_{2m}\}$ yields

$$\zeta_{D^2}(s;S^{2m-1}) \;=\; \frac{1}{\Gamma(2s)}\cdot\chi(1)\cdot\int_0^\infty t^{2s}\,\frac{e^{-(m-\frac{1}{2})t}}{\det(1_{2m}-e^{-t}\cdot 1_{2m})}\,\frac{1-e^{-t}}{t}dt$$

and therefore

$$\begin{aligned}
\zeta_{D^2}\left(s;G\backslash S^{2m-1}\right) \quad&=\quad \frac{\zeta_{D^2}(s;S^{2m-1})}{|G|}\\
+\frac{1}{|G|\cdot\Gamma(2s)}\sum_{g\in G-\{1_{2m}\}}&\chi(\varepsilon(g))\int_0^\infty t^{2s}\,\frac{e^{-(m-\frac{1}{2})t}}{\det(1_{2m}-e^{-t}g)}\,\frac{1-e^{-t}}{t}\,dt\;.
\end{aligned}$$

Since $g\in G-\{1_{2m}\}$ does not have any positive real eigenvalues the integral defines a function in s holomorphic at 0. Thus

$$\zeta_{D^2}(0;G\backslash S^{2m-1}) \;=\; \frac{\zeta_{D^2}(0;S^{2m-1})}{|G|} \;=\; 0 \tag{21}$$

by (20) and

$$\begin{aligned}
\zeta'_{D^2}(0;G\backslash S^{2m-1}) \quad&=\quad \frac{\zeta'_{D^2}(0;S^{2m-1})}{|G|}\\
+\frac{2}{|G|}\sum_{g\in G-\{1_{2m}\}}&\chi(\varepsilon(g))\int_0^\infty\frac{e^{-(m-\frac{1}{2})t}}{\det(1_{2m}-e^{-t}g)}\,\frac{1-e^{-t}}{t}\,dt
\end{aligned}$$

by (5). This implies the formula for $\mathrm{DET}(D^2;G\backslash S^{2m-1})$.

Equality of $\mathrm{DET}(D^2;G\backslash S^{2m-1})$ and $\mathrm{DET}_{\mathrm{p.t.}}(D^2;G\backslash S^{2m-1})$ follows from Proposition 2.1 and (21).

Example 2. Let us look at real projective space $M = \mathbb{RP}^{2m-1}$. In this case $G = \{1_{2m}, -1_{2m}\}$. We assume m to be odd so that M is spin and has two spin structures. They are given by $\varepsilon(-1_{2m}) = \pm\omega$ where $\omega = e_1 \cdots e_{2m}$ is the volume element considered as an element in $\mathrm{Spin}(2m) \subset \mathrm{Cl}(\mathbb{R}^{2m})$. The volume element acts by multiplication by ± 1 on the two half spinor spaces so that $\chi^+(\omega) = -\chi^-(\omega)$. Hence for both spin structures we have $\chi(\varepsilon(-1_{2m})) = 0$. It follows that

$$\mathrm{DET}(D^2; \mathbb{RP}^{2m-1}) = \sqrt{\mathrm{DET}(D^2; S^{2m-1})} = \mathrm{DET}(D; S^{2m-1}).$$

Example 3. Let us now look at the lens space $M = L(q, p_1, \ldots, p_m)$. We assume that q is odd so that M has exactly one spin structure. For

$$g = \begin{pmatrix} R(k\,p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(k\,p_m/q) \end{pmatrix}$$

we compute

$$\begin{aligned}
\chi(\varepsilon(g)) &= \sum_{\varepsilon_1, \cdots, \varepsilon_m} \exp\left(\pi i k(q+1) \sum_{j=1}^{m} \varepsilon_j p_j / q\right) \\
&= \sum_{\varepsilon_1, \cdots, \varepsilon_m} \prod_{j=1}^{m} \exp(\pi i k(q+1)\varepsilon_j p_j/q) \\
&= 2^m \prod_{j=1}^{m} \cos\left(\frac{\pi\, k\,(q+1)\, p_j}{q}\right)
\end{aligned}$$

and

$$\begin{aligned}
\det\left(1_{2m} - e^{-t}\, g\right) &= \prod_{j=1}^{m} \det\left(1_{2m} - e^{-t}\, R(k\,p_j/q)\right) \\
&= \prod_{j=1}^{m} \left(1 - 2\, e^{-t}\, \cos(2\,\pi\, k\, p_j/q) + e^{-2t}\right).
\end{aligned}$$

Now Theorem 5.1 yields

$$\mathrm{DET}(D^2; L(q, p_1, \ldots, p_m)) = \mathrm{DET}(D^2; S^{2m-1})^{1/q} \cdot \exp(A)$$

where the covering anomaly $\exp(A)$ is given by

$$A = -\frac{2^{m+1}}{q} \sum_{k=1}^{q-1} \int_0^{\infty} \frac{(1-e^{-t})\; e^{-(m-1/2)t}\; dt}{t \prod_j (1 - 2\, e^{-t}\, \cos(2\,\pi\, k\, p_j/q) + e^{-2t})} \prod_{j=1}^{m} \cos\left(\frac{\pi\, k\,(q+1)\, p_j}{q}\right).$$

Theorem 5.2 *Let* $M = G\backslash S^{2m-1}$ *be a spherical space form with spin structure given by* $\varepsilon : G \to \mathrm{Spin}(2m)$. *Then the Dirac operator on* M *has the determinant*

$$\begin{aligned}
&\mathrm{DET}\,(D; G\backslash S^{2m-1}) \\
&= \exp\left(\frac{i\pi}{|G|} \sum_{g \in G - \{1_{2m}\}} \frac{(\chi^+ - \chi^-)(\varepsilon(g))}{\det(1_{2m} - g)}\right) \cdot \sqrt{\mathrm{DET}(D^2; G\backslash S^{2m-1})}\,,
\end{aligned}$$

where $\mathrm{DET}\,(D^2; G\backslash S^{2m-1})$ *is given by Theorem 5.1 and* $\chi^{\pm}$ *are the characters of the half spinor modules of* $\mathrm{Spin}\,(2m)$.

Proof. Follows directly from (3), Theorem 3.2, and (21).

6 Spectrum and Determinant of twisted Dirac operators on S^2

In this section we compute the spectrum and determinant of certain twisted Dirac operators on the two-sphere S^2. Recall that complex line bundles over S^2 are in 1-1 correspondence with $\mathbb{Z}$ via their Chern numbers. Hence each complex line bundle is of the form $\mathcal{L}^k$ for some $k \in \mathbb{Z}$ where $c_1(\mathcal{L}) = 1$. The spinor bundle ΣS^2 decomposes into a sum of two complex line bundles, $\Sigma S^2 = \Sigma^+ S^2 \oplus \Sigma^- S^2$, the bundles of spinors of positive and of negative chirality. Since $c_1(\Sigma^\pm S^2) = \pm 1$ we have $\mathcal{L} = \Sigma^+ S^2$. The metric and the Levi-Civita connection on $\Sigma^+ S^2$ induce Hermitian metrics and metric connections on all the line bundles $\mathcal{L}^k$. Using these connections we can form the twisted Dirac operator D_k acting on sections of the bundle $\Sigma S^2 \otimes \mathcal{L}^k$.

Theorem 6.1 *The eigenvalues of the twisted Dirac operator D_k acting on sections of the spinor bundle tensored with the line bundle of Chern number k over S^2 are given by*

$$\pm\sqrt{j(|k|+j)}, \qquad j = 0, 1, 2, \dots$$

with multiplicity

$$2j + |k|.$$

Proof. We start by computing the relevant curvatures. The curvature tensor $R^{\Sigma S^2}$ of the spinor bundle can be expressed in terms of the curvature tensor R^{TS^2} of the tangent bundle by

$$R^{\Sigma S^2}(X,Y) = \frac{1}{4}\sum_{i,j=1}^{2} \left\langle R^{TS^2}(X,Y)e_i, e_j \right\rangle e_i e_j \cdot$$

where e_1, e_2 is an orthonormal tangent frame acting on spinors via Clifford multiplication. Since in our case S^2 is of constant sectional curvature 1 we have

$$R^{TS^2}(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

and therefore

$$R^{\Sigma S^2}(X,Y) = \frac{1}{4}(YX - XY)\cdot$$

Hence

$$R^{\Sigma S^2}(e_1, e_2) = -\frac{1}{2} e_1 \cdot e_2.$$

Since the area element $e_1 \cdot e_2$ acts on $\Sigma^\pm S^2$ by multiplication by $\pm i$ we get

$$R^{\mathcal{L}}(e_1, e_2) = R^{\Sigma^+ S^2}(e_1, e_2) = -\frac{i}{2}$$

and thus

$$R^{\mathcal{L}^m}(e_1, e_2) = -\frac{im}{2}.$$

The decomposition $\Sigma S^2 = \mathcal{L} \oplus \mathcal{L}^{-1}$ yields the parallel decomposition $\Sigma S^2 \otimes \mathcal{L}^k = \mathcal{L}^{k+1} \oplus \mathcal{L}^{k-1}$ with respect to which the curvature tensor has the form

$$R^{\Sigma S^2 \otimes \mathcal{L}^k}(e_1, e_2) = -\frac{i}{2}\begin{pmatrix} k+1 & 0 \\ 0 & k-1 \end{pmatrix}.$$

The Dirac operator interchanges the chirality of spinors and hence has the form

$$D_k = \begin{pmatrix} 0 & D_k^+ \\ D_k^- & 0 \end{pmatrix}$$

where $D_k^- = (D_k^+)^*$. Denote the connection Laplace operator on $\mathcal{L}^m$ by $\nabla^*\nabla =: \Delta_m$. The curvature endomorphism in the Weitzenböck formula for D_k^2 is given by

$$\begin{aligned}\mathcal{K}_k &= e_1 \cdot e_2 \cdot R^{\Sigma S^2 \otimes \mathcal{L}^k}(e_1, e_2) \\ &= -\frac{i}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} k+1 & 0 \\ 0 & k-1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} k+1 & 0 \\ 0 & -k+1 \end{pmatrix}.\end{aligned}$$

The Weitzenböck formula $D_k^2 = \nabla^*\nabla + \mathcal{K}_k$ now says

$$D_k^+ D_k^- = \Delta_{k+1} + \frac{k+1}{2}, \tag{22}$$

$$D_k^- D_k^+ = \Delta_{k-1} + \frac{-k+1}{2}. \tag{23}$$

Taking the difference of (22) and (23) with $k+2$ instead of k we obtain

$$D_k^+ D_k^- = D_{k+2}^- D_{k+2}^+ + k + 1. \tag{24}$$

Since $D_k^+ D_k^- = (D_k^-)^* D_k^-$ and $D_k^- D_k^+ = (D_k^+)^* D_k^+$ are non-negative operators they do not have negative eigenvalues. Denote the positive eigenvalues by $\lambda_j(D_k^+ D_k^-)$ and $\lambda_j(D_k^- D_k^+)$, $j = 1, 2, 3, \ldots$, and their multiplicities by $\mu_j(D_k^+ D_k^-)$ and $\mu_j(D_k^- D_k^+)$ respectively. Write $\lambda_0(D_k^+ D_k^-) = \lambda_0(D_k^- D_k^+) = 0$ and let $\mu_0(D_k^+ D_k^-)$ and $\mu_0(D_k^- D_k^+)$ be the multiplicities of the eigenvalue 0. Here $\mu_0(D_k^+ D_k^-) = 0$ or $\mu_0(D_k^- D_k^+) = 0$ is not excluded.

From now on assume $k \geq 0$. The case of negative k can be treated similarly by interchanging the roles of D_k^+ and D_k^-. For non-negative k equation (22) shows

$$\mu_0(D_k^+ D_k^-) = 0. \tag{25}$$

The Atiyah-Singer index formula yields

$$\mu_0(D_k^- D_k^+) - \mu_0(D_k^+ D_k^-) = \mathrm{ind}(D_k^+) = c_1(\mathcal{L}^k) = k$$

and thus

$$\mu_0(D_k^- D_k^+) = k. \tag{26}$$

Since D_k^+ intertwines the operators $D_k^- D_k^+$ and $D_k^+ D_k^-$ it induces isomorphisms on the eigenspaces for non-zero eigenvalues. Hence for $j \geq 1$

$$\lambda_j(D_k^+ D_k^-) = \lambda_j(D_k^- D_k^+) \quad \text{and} \quad \mu_j(D_k^+ D_k^-) = \mu_j(D_k^- D_k^+). \tag{27}$$

Combining equations (24) and (27) yields for $j \geq 1$

$$\lambda_j(D_k^- D_k^+) = \lambda_{j-1}(D_{k+2}^- D_{k+2}^+) + k + 1 \text{ and } \mu_j(D_k^- D_k^+) = \mu_{j-1}(D_{k+2}^- D_{k+2}^+). \tag{28}$$

The parameter shift from j to $j-1$ is due to the fact that 0 is counted as an eigenvalue (of multiplicity 0) for $D_k^+ D_k^-$. Using (28) inductively j times we get

$$\lambda_j(D_k^- D_k^+) = \lambda_0(D_{k+2j}^- D_{k+2j}^+) + \sum_{m=1}^{j}(k + 2m - 1) = j(k+j)$$

and by (26)

$$\mu_j(D_k^- D_k^+) = \mu_0(D_{k+2j}^- D_{k+2j}^+) = k + 2j.$$

We have computed the spectrum of $D_k^2 = \begin{pmatrix} D_k^+ D_k^- & 0 \\ 0 & D_k^- D_k^+ \end{pmatrix}$, namely 0 is an eigenvalue of multiplicity k and $j(k+j)$ has multiplicity $2(k+2j)$, $j = 1, 2, 3, \ldots$.

It remains to observe that the spectrum of D_k is symmetric about 0 because for each eigenspinor of D_k of the form $\psi = \psi_+ + \psi_-$ with respect to the splitting $\Sigma S^2 \otimes \mathcal{L}^k = \Sigma^+ S^2 \otimes \mathcal{L}^k \oplus \Sigma^- S^2 \otimes \mathcal{L}^k$ the spinor $\hat{\psi} = \psi_+ - \psi_-$ is an eigenspinor for the opposite eigenvalue. Taking square roots proves the theorem.

For $k = 0$ Theorem 6.1 gives the spectrum of the classical Dirac operator which we used in the previous sections. Similarly, for $k = 1$ we recover the spectrum of the Laplace operator acting on functions $\Delta = D_1^- D_1^+$.

Theorem 6.1 can also be proved by trivialising the twisted spinor bundle over S^2 minus a point and then solving the eigenvalue equation explicitly using spin-weighted spherical harmonics. See [16, Sec. 3.1] for this approach.

Knowing the eigenvalues of D_k explicitly we can compute its determinant.

Theorem 6.2 *The determinant of the twisted Dirac operator D_k acting on sections of the spinor bundle tensored with the line bundle of Chern number k over S^2 is given by*

$$\mathrm{DET}(D_k; S^2) = \exp\left(i\frac{\pi}{2}(-|k| - \frac{1}{3})\right) e^{-4\,\zeta_R'(-1) + k^2/2} \cdot \prod_{m=1}^{|k|} m^{|k|-2m},$$

the determinant of its square is

$$\mathrm{DET}(D_k^2; S^2) = e^{-8\,\zeta_R'(-1) + k^2} \cdot \prod_{m=1}^{|k|} m^{2|k|-4m}$$

and the proper time regularized determinant

$$\mathrm{DET}_{\mathrm{p.t.}}(D_k^2; S^2) = e^{-8\,\zeta_R'(-1) + k^2 + \Gamma'(1)(|k| + \frac{1}{3})} \cdot \prod_{m=1}^{|k|} m^{2|k|-4m}.$$

Proof. By Theorem 6.1 we obtain for the ζ-function

$$\begin{aligned}\zeta_{D_k^2}(s) &= 2\sum_{j=1}^{\infty} \frac{2j + |k|}{j^s (j+|k|)^s} \\ &= 2\sum_{j=1}^{\infty} (j^{-s}(j+|k|)^{1-s} + j^{1-s}(j+|k|)^{-s}).\end{aligned}$$

For $j > |k|$ we can expand $(1 + \frac{|k|}{j})^\alpha$ into a binomial series and we get

$$\begin{aligned}\zeta_{D_k^2}(s) \quad &= \\ &= 2\sum_{j=1}^{|k|} \frac{2j+|k|}{j^s(j+|k|)^s} + 2\sum_{j=|k|+1}^{\infty} j^{1-2s} \sum_{i=0}^{\infty} \left[\binom{1-s}{i} + \binom{-s}{i}\right] \left(\frac{|k|}{j}\right)^i \\ &= 2\sum_{j=1}^{|k|} \frac{2j+|k|}{j^s(j+|k|)^s} + 2\sum_{i=0}^{\infty} \left[\binom{1-s}{i} + \binom{-s}{i}\right] |k|^i \zeta_H(2s+i-1, |k|+1)\end{aligned}$$

Now we partly follow a similar calculation of Weisberger [21, Appendix C]. For the coefficients

$$d_i(s) := \binom{1-s}{i} + \binom{-s}{i}$$

we have

$$\begin{aligned} d_0(s) &= 2, \\ d_1(s) &= -2s+1, \\ d_2(s) &= s^2, \\ d_i(s) &= (-1)^i \frac{i-2}{i(i-1)} s + \mathrm{O}(s^2), \qquad i \geq 3. \end{aligned}$$

Hence

$$\begin{aligned} \zeta_{D_k^2}(0) &= 2\sum_{j=1}^{|k|}(2j+|k|) + 2\left(2\zeta_H(-1,|k|+1) + |k|\zeta_H(0,|k|+1)\right) \\ &= 4\zeta_R(-1) + 2|k|\zeta_R(0) \\ &= -\frac{1}{3} - |k| \end{aligned} \tag{29}$$

and since $\zeta_H(s,|k|+1)$ has only one pole of first order at $s=1$ with residue 1 we have

$$\begin{aligned} \zeta'_{D_k^2}(0) \quad &= \\ = -2\sum_{j=1}^{|k|} &(2j+|k|)(\ln j + \ln(j+|k|)) + 8\zeta'_H(-1,|k|+1) + 4|k|\zeta'_H(0,|k|+1) \\ &-4|k|\zeta_H(0,|k|+1) + |k|^2 + 2\sum_{i=3}^{\infty}(-1)^i \frac{i-2}{i(i-1)}|k|^i \zeta_H(i-1,|k|+1) \\ = -2\sum_{j=1}^{|k|} &(2j+|k|)(\ln j + \ln(j+|k|)) + 8\zeta'_H(-1,|k|+1) + 4|k|\zeta'_H(0,|k|+1) \\ &+2|k| + 5|k|^2 + 2\sum_{i=3}^{\infty}(-1)^i \frac{i-2}{i(i-1)}|k|^i \zeta_H(i-1,|k|+1). \end{aligned} \tag{30}$$

For the evaluation of the last sum we use the Mellin transform of the Hurwitz ζ-function

$$\zeta_H(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at}(1-e^{-t})^{-1} dt. \tag{31}$$

This yields

$$\begin{aligned} I &:= \sum_{i=3}^{\infty}(-1)^i \frac{i-2}{i(i-1)}|k|^i \zeta_H(i-1,|k|+1) \\ &= \sum_{i=3}^{\infty}(-1)^i \frac{i-2}{i(i-1)}|k|^i \frac{1}{(i-2)!}\int_0^\infty t^{i-2} e^{-(|k|+1)t}(1-e^{-t})^{-1} dt \\ &= |k| \int_0^\infty t^{-1}(-e^{-(2|k|+1)t} + e^{-(|k|+1)t}(1-|k|t))(1-e^{-t})^{-1} dt \\ &\quad -2\int_0^\infty t^{-2}(e^{-(2|k|+1)t} - e^{-(|k|+1)t}(1-|k|t + \frac{1}{2}(|k|t)^2))(1-e^{-t})^{-1} dt. \end{aligned}$$

To compute these integrals we multiply the integrand by t^s, evaluate the integral using (31) and then let $s \to 0$.

$$\begin{aligned}I = \lim_{s\to 0}\{&|k|\Gamma(s)(-\zeta_H(s,2|k|+1)+\zeta_H(s,|k|+1))\\&-|k|^2\Gamma(s+1)\zeta_H(s+1,|k|+1)-2[\Gamma(s-1)(\zeta_H(s-1,2|k|+1)\\&-\zeta_H(s-1,|k|+1))+|k|\Gamma(s)\zeta_H(s,|k|+1)\\&-\frac{1}{2}|k|^2\Gamma(s+1)\zeta_H(s+1,|k|+1)]\}\\=\lim_{s\to 0}&\{|k|\Gamma(s)(-\zeta_H(s,2|k|+1)-\zeta_H(s,|k|+1))+2\Gamma(s-1)h(s-1)\}\end{aligned}$$

with the definition

$$h(s) \quad := \quad \sum_{j=1}^{|k|}(j+|k|)^{-s} \quad = \quad \zeta_H(s,|k|+1)-\zeta_H(s,2|k|+1).$$

Using $-2\zeta_H(0,|k|+1)=1+2|k|$, $h(0)=|k|$, and $\mathrm{Res}_{s=-l}\Gamma(s)=\frac{(-1)^l}{l!}$ we conclude

$$\begin{aligned}I &=\\&=\lim_{s\to 0}\left\{|k|s\Gamma(s)\left(-2\frac{\zeta_H(s,|k|+1)-\zeta_H(0,|k|+1)}{s}+\frac{h(s)-h(0)}{s}+\frac{1+3|k|}{s}\right)\right.\\&\quad\left.+2\Gamma(s-1)h(s-1)\right\}\\&=-2|k|\zeta_H'(0,|k|+1)+|k|h'(0)\\&\quad+\lim_{s\to 0}\{2\Gamma(s-1)[h(s-1)+(s-1)\frac{|k|}{2}(1+3|k|)]\}\\&=-2|k|\zeta_H'(0,|k|+1)+|k|h'(0)-2h'(-1)-|k|(1+3|k|)\\&=-2|k|\zeta_H'(0,|k|+1)-|k|\sum_{j=1}^{|k|}\ln(j+|k|)+2\sum_{j=1}^{|k|}\ln(j+|k|)(j+|k|)\\&\quad-|k|(1+3|k|)\\&=-2|k|\zeta_H'(0,|k|+1)+\sum_{j=1}^{|k|}\ln(j+|k|)(2j+|k|)-|k|(1+3|k|).\end{aligned}$$

Inserting this into (30) we get

$$\begin{aligned}\zeta'_{D_k^2}(0)&=-2\sum_{j=1}^{|k|}(2j+|k|)\ln j+8\zeta_H'(-1,|k|+1)+2|k|+5|k|^2\\&\quad-2|k|(1+3|k|)\\&=8\zeta_R'(-1)-|k|^2+2\sum_{j=1}^{|k|}(2j-|k|)\ln j,\end{aligned}$$

therefore

$$\mathrm{DET}(D_k^2;S^2)=e^{-8\zeta_R'(-1)+k^2}\cdot\prod_{m=1}^{|k|}m^{2|k|-4m}.$$

The formula for $\mathrm{DET}(D_k;S^2)$ now follows from (29) and from the vanishing of the η-invariant. Proposition 2.1 yields the formula for $\mathrm{DET}_{\mathrm{p.t.}}(D_k^2;S^2)$.

Remark 1. The case $k=0$ yields Theorem 4.1, 4.2 and 4.3 for $n=2$. Since the Laplace-Beltrami operator acting on functions is given by $\Delta = D_1^-D_1^+$ it has the same spectrum as D_1^2 except that all non-zero eigenvalues have only half the multiplicity. Hence $\mathrm{DET}(\Delta;S^2)=\sqrt{\mathrm{DET}(D_1^2;S^2)}$ and we get (compare [4, Thm. 8.1])