A K-theoretic Proof of Boutet de Monvel's Index Theorem

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Abstract

Let $\mathfrak{A}$ be the C^*-closure of the algebra of all polyhomogeneous operators of order and class zero in Boutet de Monvel's calculus on a connected manifold X with non-empty boundary ∂X. We derive a natural short exact sequence

$$0 \to K_i(C(X)) \to K_i(\mathfrak{A}/\mathfrak{K}) \to K_{1-i}(C_0(T^*X^\circ)) \to 0,$$

$i = 0, 1$, which splits in a not necessarily natural way. Here $\mathfrak{K}$ is the ideal of compact operators, and T^*X° is the cotangent bundle of the interior of X. This strengthens a result of Melo, Nest and Schrohe in [6].

We also give a K-theoretic proof of the fact that the composition of the topological index with the mapping $K_1(\mathfrak{A}/\mathfrak{K}) \to K_0(C_0(T^*X^\circ))$ in the above sequence gives the Fredholm-index homomorphism.

That the Fredholm index for $\mathfrak{A}$ factors through $K(T^*X^\circ)$ was first proven by Boutet de Monvel [3].

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1 Introduction

In [6], Melo, Nest and Schrohe compute the K-theory of the C^*-closure of the Boutet de Monvel algebra $\mathfrak{A}$ on a compact connected manifold X with nonempty boundary ∂X. For all definitions and for preliminary results, we refer to that paper. In particular, we closely follow the notation established there without further reference.

In that paper the short exact sequence

$$0 \to \ker(r_*^i) \oplus K_{1-i}(C_0(T^*X^\circ)) \to K_i(\mathfrak{A}/\mathfrak{K}) \to \operatorname{im}(r_*^i) \to 0,\ i = 0, 1,$$

is derived [6, Theorem 3], where $\mathfrak{K}$ denotes the ideal of compact operators on the Hilbert space $\mathfrak{H}$ onto which $\mathfrak{A}$ acts, $\mathfrak{H} = L^2(X) \oplus H^{-\frac{1}{2}}(\partial X)$. If the K-groups of $C(\partial X)$ have no torsion, this becomes

$$K_i(\mathfrak{A}/\mathfrak{K}) \cong K_i(C(X)) \oplus K_{1-i}(C_0(T^*X^\circ)).$$

There it is also shown that this splitting holds for $i = 1$, even without this assumption on the K-theory of ∂X. For that, the authors use the fact [3, Theorem 5.21], that the Fredholm-index homomorphism for $\mathfrak{A}$ is the composition of the topological index $\chi : K_0(C_0(T^*X^\circ)) \to \mathbb{Z}$, with a homomorphism from $K_1(\mathfrak{A}/\mathfrak{K})$ to $K_0(C_0(T^*X^\circ))$.

In this article, we want to improve those results by giving a K-theoretic proof of Theorem 1, below. Our proof uses the descriptions given in [6] of the kernel and of the image of the boundary principal symbol, but makes no use of Boutet de Monvel's index theorem. In Theorem 2, we show, instead, that the existence of the homomorphism from $K_1(\mathfrak{A}/\mathfrak{K})$ to $K_0(C_0(T^*X^\circ))$, explicitly constructed in [3] and denoted `ind` in [6], actually follows from our K-theory arguments.

We also address the question, somewhat neglected in [6], of the naturality of these sequences and isomorphisms.

Theorem 1. *We have natural short exact sequences*

$$0 \to K_i(C(X)) \longrightarrow K_i(\mathfrak{A}/\mathfrak{K}) \xrightarrow{p} K_{i-1}(C_0(T^*X^\circ)) \to 0, \tag{1}$$

which split; i.e. we have (not necessarily natural) isomorphisms

$$K_i(\mathfrak{A}/\mathfrak{K}) \cong K_i(C(X)) \oplus K_{1-i}(C_0(T^*X^\circ)), \tag{2}$$

without any further conditions.

Theorem 2. *The index of a Fredholm operator A in $\mathfrak{A}$ (or of some $k \times k$ matrix A acting as a Fredholm operator on the sum of k copies of $\mathfrak{H}$) is given by*

$$\operatorname{ind} A = \chi(p([A])), \tag{3}$$

where $[A]$ is K_1-class of the canonical projection of A in $\mathfrak{A}/\mathfrak{K}$, p is the mapping in (1) and

$$\chi : K_0(C_0(T^*X^\circ)) \cong K^0(T^*X^\circ)) \to K^0(pt) \cong \mathbb{Z}$$

is the topological index. Moreover, the mappings p and `ind` coincide.

Remark 3. In Section 4, we will replace the elaborate calculations of Boutet de Monvel by our Theorem 2 in the derivation of Fedosov's index formula [4, Theorem 2.4]

$$\operatorname{ind} A = \int_{T^*X} \operatorname{ch} d_i(A)\mathcal{T}(X) + \int_{T^*\partial X} \operatorname{ch}' d_b(A)\mathcal{T}(X). \tag{4}$$

Here $\mathcal{T}(X)$ is the differential form given by the Todd class of the tangent bundle, and $\operatorname{ch} d_i(A)$ and $\operatorname{ch}' d_b(A)$ are analogs of the Chern character for the interior and the boundary symbol. It should be noted that we cannot replace Fedosov's arguments.

2 K-theory Preliminaries

Definition 1. Let A be a C^*-algebra. The cone over A is the C^*-algebra $CA := \{f : [0,1] \to A \mid f(1) = 0\}$.

Since CA is a contractible C^*-algebra, its K-theory vanishes. The suspension of A is given by $SA := \{f \in CA \mid f(0) = 0\}$.

Definition 2. If $f : B \to A$ is a C^*-algebra homomorphism, the mapping cone Cf is defined to be $Cf := \{(b, \phi) \in B \oplus CA;\ f(b) = \phi(0)\}$.

Projection onto B defines a short exact sequence

$$0 \longrightarrow SA \xrightarrow{\hat{\imath}} Cf \xrightarrow{q} B \longrightarrow 0. \tag{5}$$

The assignment of this exact sequence to each C*-algebra homomorphism f defines a functor between the corresponding categories (whose morphisms consist of commutative diagrams of homomorphisms or of exact sequences, respectively). This functor is exact; i.e. we have:

Lemma 1. *Assume that the following is a commutative diagram of short exact sequences of C^*-algebras:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "f"] & B' \arrow[r] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

Then we get an induced commutative grid of short exact sequences of C^*-algebras

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & B' \arrow[r] \arrow[u] & C' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Cf \arrow[r] \arrow[u] & Cg \arrow[r] \arrow[u] & Ch \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & SA \arrow[r] \arrow[u] & SB \arrow[r] \arrow[u] & SC \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Lemma 1 can be proven by a diagram chase, using that the maps $CB \to CC$ and $SB \to SC$ are surjective (the exactness of S is proven in [7, Proposition 10.1.2]).

Lemma 2. *The exact sequence* (5) *induces six-term cyclic exact sequences in K-theory, whose connecting mappings $K_i(B) \to K_{1-i}(SA)$ become, under the canonical isomorphisms $K_{1-i}(SA) \xrightarrow{\cong} K_i(A)$, the mappings induced by f.*

More precisely, this lemma states that the two diagrams

$$\begin{tikzcd}
K_1(B) \arrow[r, "\delta_0"] & K_0(SA) \\
K_1(B) \arrow[r, "f_*"] \arrow[u, "="] & K_1(A) \arrow[u, "\Theta_A"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
K_0(B) \arrow[r, "\delta_1"] & K_1(SA) \\
K_0(B) \arrow[r, "f_*"] \arrow[u, "="] & K_0(A) \arrow[u, "\beta_A"]
\end{tikzcd} \tag{6}$$

commute, where δ_0 and δ_1 denote, respectively, the index and the exponential mappings [7, 9.1.3 and 12.1.1] induced by (5), Θ_A is the isomorphism defined in

[7, 10.1.3], and β_A is the Bott isomorphism [7, 11.1.1]. This follows by applying Lemma 1 and naturality of the long exact sequences in K-theory to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(f) & \longrightarrow & B & \xrightarrow{f} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0.
\end{array}$$

Lemma 3. *If $f : B \to A$ is a surjective C^*-homomorphism, then the mapping $j : \ker f \ni x \mapsto (x, 0) \in Cf$ induces a K-theory isomorphism, which fits into the commutative diagram*

$$\begin{array}{ccccccccccc}
\longrightarrow & K_{i+1}(B) & \longrightarrow & K_i(SA) & \xrightarrow{\hat\imath_*} & K_i(Cf) & \xrightarrow{q_*} & K_i(B) & \longrightarrow \\
 & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle \delta_{i+1}} & & \uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle =} & \\
\longrightarrow & K_{i+1}(B) & \xrightarrow{f_*} & K_{i+1}(A) & \longrightarrow & K_i(\ker f) & \longrightarrow & K_i(B) & \longrightarrow
\end{array} \tag{7}$$

where the upper row is the cyclic exact sequence induced by (5)*, and the lower one is that induced by*

$$0 \longrightarrow \ker f \longrightarrow B \xrightarrow{f} A \longrightarrow 0. \tag{8}$$

Proof. This result is certainly well known. For the sake of completeness, and since we did not find a convenient reference, we sketch the arguments. One proves that j_* is a K-theory isomorphism using that $K_*(CA) = 0$ in the cyclic exact sequence associated to the short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{j} Cf \longrightarrow CA \longrightarrow 0$$

induced by projection of Cf onto the second coordinate. The boundary map δ_{i+1} is well known to be an isomorphism. It remains to establish commutativity. The commutativity of the left rectangle is part of Lemma 2. The right rectangle commutes by naturality of the K-theory functor, since $q \circ j = (i\colon \ker f \hookrightarrow B)$. The argument for the middle square is a little more involved. Observe that $j_*^{-1}\hat\imath_*\delta_{i+1}$ defines another homomorphism $K_{i+1}(A) \to K_i(\ker f)$ which is, by the naturality of all constructions, natural and makes the K-theory sequence of the short exact sequence of C^*-algebras exact.

However, homomorphisms with this properties are defined uniquely (up to a universal sign) [8, Exercise 9.F], therefore the diagram is commutative up to this sign. The special exact sequence $0 \to SA \to CA \to A \to 0$ shows that this sign is $+1$. □

3 Proof of the splitting result

We start applying Lemma 1 to the diagram [6, (34)]:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathfrak{I}/\mathfrak{K} & \longrightarrow & \mathfrak{A}/\mathfrak{K} & \xrightarrow{\pi} & \mathfrak{A}/\mathfrak{I} & \longrightarrow 0 \\
 & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle b} & \\
0 \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) & \longrightarrow 0
\end{array} \tag{9}$$

Here $\mathfrak{I}$ is the C^*-closure of all elements of $\mathfrak{A}$ supported in the interior; m is the isometric *-homomorphism that maps $f \in C(X)$ to the class of $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$; g denotes the restriction to ∂X of f. Recall that b was defined by mapping a function g to the image, under the boundary symbol map γ, of $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$, where f is an arbitrary function whose restriction to ∂X is g. We do not distinguish between the isomorphic C^*-algebras $\mathfrak{A}/\mathfrak{I}$ and the image of γ. We obtain

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \longrightarrow & C(\partial X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Cm_{X^\circ} & \longrightarrow & Cm & \longrightarrow & Cb & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S(\mathfrak{I}/\mathfrak{K}) & \longrightarrow & S(\mathfrak{A}/\mathfrak{K}) & \longrightarrow & S(\mathfrak{A}/\mathfrak{I}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{10}$$

These six short exact sequences induce now six six-term cyclic exact sequences in K-theory which we want to analyze next. By [6, Corollary 8], b induces an isomorphism in K-theory. From Lemma 2 and the cyclic exact sequence of $0 \to S(\mathfrak{A}/\mathfrak{I}) \to Cb \to C(\partial X) \to 0$ we therefore conclude that $K_*(Cb) = 0$. From this in turn we deduce, using the cyclic exact sequence of $0 \to Cm_{X^\circ} \to Cm \to Cb \to 0$, that $Cm_{X^\circ} \to C_m$ induces an isomorphism in K-theory.

We therefore get the following commutative diagram of cyclic exact sequences of K-theory groups, again using Lemma 2 and the natural isomorphisms $K_{1-i}(SA) \cong K_i(A)$.

$$\begin{array}{ccccccccccc}
\longrightarrow & K_i(Cm) & \longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
 & \uparrow\cong & & \uparrow & & i\uparrow & & \phi\uparrow\cong & \\
\longrightarrow & K_i(Cm_{X^\circ}) & \longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_*} & K_i(\mathfrak{I}/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_{X^\circ}) & \longrightarrow
\end{array} \tag{11}$$

By [6, Theorem 1], $\mathfrak{I}/\mathfrak{K} \cong C_0(S^*X^\circ)$, via the principal symbol map. Moreover, if, as we assume throughout, X is connected and $\partial X \neq \emptyset$, it is also shown that m has a split s (which is not necessarily natural). Consequently, α also has a split s'. Define now $s'' := i \circ s' \circ \phi^{-1} \colon K_{1-i}(Cm) \to K_i(\mathfrak{A}/\mathfrak{K})$. An easy diagram chase shows that s'' is a split of β. Consequently, our long exact sequence yields natural short exact sequences

$$0 \to K_i(C(X)) \to K_i(\mathfrak{A}/\mathfrak{K}) \to K_{1-i}(Cm) \to 0,$$

which have a (not necessarily natural) split.

It remains to identify $K_{1-i}(Cm) \cong K_{1-i}(Cm_{X^\circ})$. For this, recall the natural short exact sequence for the ball – or disc – completion of the cotangent bundle, extended to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(T^*X^\circ) & \longrightarrow & C_0(B^*X^\circ) & \xrightarrow{r} & C_0(S^*X^\circ) & \longrightarrow & 0 \\
 & & & & \pi^*r_0 \big\uparrow \sim & & \big\uparrow = & & \\
 & & & & C_0(B^*X^\circ) & \xrightarrow{\pi^*r_0} & C_0(S^*X^\circ) & & \\
 & & & & r_0 \big\downarrow \sim & & \big\downarrow = & & \\
 & & & & C_0(X^\circ) & \xrightarrow{m_{X^\circ}} & C_0(S^*X^\circ) & &
\end{array} \tag{12}$$

Here, π^* denotes pull back from the base to the total space of the bundle, and r and r_0 denote restriction to the boundary of the disc bundle, or the zero section of the disc bundle, respectively (depending on the context); $\sim$ denotes the well known homotopy equivalences of C^*-algebras.

We get induced short exact mapping cone sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & Cr & \longrightarrow & C_0(B^*X^\circ) & \longrightarrow & 0 \\
 & & \big\uparrow = & & \big\uparrow (\pi^*r_0)_* & & \sim \big\uparrow \pi^*r_0 & & \\
0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & C(\pi^*r_0) & \longrightarrow & C_0(B^*X^\circ) & \longrightarrow & 0 \\
 & & \big\downarrow = & & \sim \big\downarrow (r_0)_* & & \sim \big\downarrow r_0 & & \\
0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & Cm_{X^\circ} & \longrightarrow & C_0(X^\circ) & \longrightarrow & 0
\end{array} \tag{13}$$

The corresponding cyclic exact K-theory sequences together with the 5-lemma imply that the induced maps between the mapping cones induce isomorphisms in K-theory.

Finally, since r is surjective and $\ker r = C_0(T^*X^\circ)$, Lemma 3 yields the commutative digram

$$\begin{array}{ccc}
K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(Cr) \\
\big\uparrow & & \big\uparrow j_* \\
K_1(C_0(S^*X^\circ)) & \xrightarrow{\delta} & K_0(C_0(T^*X^\circ))
\end{array} , \tag{14}$$

where the lower horizontal arrow is the index mapping for the first row in (12), and the upper horizontal is induced by the first row in (13).

The composition of all these maps gives a natural way to identify $K_i(Cm)$ with $K_i(C_0(T^*X^\circ))$. This already finishes the proof of Theorem 1. A more detailed explanation of this step will be needed below, in the proof of Theorem 2.

4 Proof of the index results

Let us consider the following commutative diagram

$$
\begin{array}{ccccc}
K_1(C(X)) & \xrightarrow{m_*} & K_1(\mathfrak{A}/\mathfrak{K}) & \longrightarrow & K_0(Cm) \\
 & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle\cong} \\
K_1(C_0(X^\circ)) & \xrightarrow{m_{X^\circ *}} & K_1(\mathfrak{I}/\mathfrak{K}) & \longrightarrow & K_0(Cm_{X^\circ}) \\
 & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
 & & K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(C(\pi^* r_0)) \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle\cong} \\
 & & K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(Cr) \\
 & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
 & & K_1(C_0(S^*X^\circ)) & \xrightarrow{\delta} & K_0(C_0(T^*X^\circ)) \\
 & & & & \downarrow{\scriptstyle\chi} \\
 & & & & \mathbb{Z}
\end{array}
\tag{15}
$$

where the first two rows are portions of (11). The second, third and fourth rows in (15) are portions of the cyclic sequences associated to (13) (notice that, if we use the isomorphism $\mathfrak{I}/\mathfrak{K} \cong C_0(S^*X^\circ)$ as an identification, then the first column in (10) is equal to the last row in (13)), while the fourth and fifth rows are just (14).

The map p in (1) is the composition of all the maps (reverting arrows of isomorphisms when necessary) in the right column in (15), except χ, with the map from $K_1(\mathfrak{A}/\mathfrak{K})$ to $K_0(Cm)$ in the first row. To prove that $\chi \circ p$ and the Fredholm index are equal on $K_1(\mathfrak{A}/\mathfrak{K})$, it is enough to show that they are equal on the image of m_* and on the image of i_* ($K_1(\mathfrak{A}/\mathfrak{K})$ is the sum of these two images, because of how the section for β in (11) was obtained). On the image of m_*, both are zero, since any Fredholm multiplication-operator is also invertible and since the first row in (15) is exact. The commutativity of (15) then shows that all we have to prove is that $\chi \circ \delta$ is the Fredholm index.

For that, let $2X$ denote the boundaryless double of X, and let Ψ denote the C^*-closure of the algebra of all classical pseudodifferential operators of order 0 on $2X$ in the algebra of all bounded operators on $L^2(2X)$. The zero extension on the orthogonal complement of $L^2(X)$ in $L^2(2X)$ defines a *-homomorphism $\mathfrak{I}_{11} \to \Psi$, where $\mathfrak{I}_{11}$ denotes the ideal of all matrices in $\mathfrak{I}$ upper-left corner of $\mathfrak{I}$. That gives us a commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \mathfrak{I}_{11} & \longrightarrow & \mathfrak{I}_{11}/\mathfrak{K} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \Psi & \longrightarrow & \Psi/\mathfrak{K} & \longrightarrow & 0,
\end{array}
\tag{16}
$$

where we have denoted by the same symbol the compact ideal on $L^2(X)$ and on $L^2(2X)$. The canonical injection of $\mathfrak{I}_{11}$ into $\mathfrak{I}$ induces an isomorphism between $\mathfrak{I}_{11}/\mathfrak{K}$ and $\mathfrak{I}/\mathfrak{K}$ (see comments right before Theorem 1 in [6]). That isomorphism and the naturality of the index mapping for (16) then imply that

$$\begin{array}{ccc} K_1(\mathfrak{I}/\mathfrak{K}) & \longrightarrow & K_0(\mathfrak{K}) \cong \mathbb{Z} \\ \downarrow{\iota_*} & & \downarrow{=} \\ K_1(\Psi/\mathfrak{K}) & \longrightarrow & K_0(\mathfrak{K}) \cong \mathbb{Z} \end{array} \tag{17}$$

commutes, where the horizontal arrows are the Fredholm-index homomorphisms for $\mathfrak{I}$ and for Ψ.

For any closed manifold, the principal symbol induces an isomorphism between $\Psi/\mathfrak{K}$ and the continuous functions on the co-sphere bundle. This follows from the classical norm-estimate for the norm, modulo compacts, of a pseudo-differential operator [5, Theorem A.4]. We therefore have $K_1(\Psi/\mathfrak{K}) \cong K_1(C(S^*(2X)))$. Modulo this isomorphism, the Atiyah-Singer index theorem [1] states that the Fredholm-index homomorphism for Ψ is the composition of the topological index $\chi^{2X} : K_0(C_0(T^*(2X)) \to \mathbb{Z}$ with the index mapping for the exact sequence

$$0 \to C_0(T^*(2X)) \to C(B^*(2X)) \to C(S^*(2X)) \to 0$$

(see [6, Proposition 15] for a proof that the classical *difference bundle* construction indeed gives the C*-algebra K-theory index mapping for this sequence).

Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(T^*X^\circ) & \longrightarrow & C_0(B^*X^\circ) & \longrightarrow & C_0(S^*X^\circ) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\iota} & & \\ 0 & \longrightarrow & C_0(T^*(2X)) & \longrightarrow & C(B^*(2X)) & \longrightarrow & C(S^*(2X)) & \longrightarrow & 0. \end{array} \tag{18}$$

The naturality of the index mapping shows that the following diagram commutes:

$$\begin{array}{ccccc} K_1(C_0(S^*X^\circ)) & \xrightarrow{\delta} & K_0(C_0(T^*X^\circ)) & & \\ \downarrow{\iota_*} & & \downarrow{\tilde{\iota}} & & \\ K_1(C(S^*(2X))) & \longrightarrow & K_0(C_0(T^*(2X))) & \xrightarrow{\chi^{2X}} & \mathbb{Z}. \end{array} \tag{19}$$

The Atiyah-Singer index theorem and the commutativity of (17) imply that the composition of the two lower horizontal and the left vertical arrows in (19) gives the Fredholm index. This shows that $\chi^{2X} \circ \tilde{\iota} \circ \delta$ is the Fredholm index for $\mathfrak{I}$. This proves (3) since, by definition, $\chi = \chi^{2X} \circ \tilde{\iota}$.

To show the last statement, $p = \texttt{ind}$, let us look again at (15). As before, it is enough to prove that $\texttt{ind} \circ i_* = p \circ i_*$ and $\texttt{ind} \circ m_* = p \circ m_*$. Our diagram gives $p \circ i_* = \delta$ and $p \circ m_* = 0$, while $\texttt{ind} \circ i_* = \delta$ and $\texttt{ind} \circ m_* = 0$ are proven in [6, Lemmas 16 and 17]. This shows Theorem 2.

To establish Fedosov's index formula (4), observe first that Fedosov's proof of "homotopy invariance" in [4, Proof of Theorem 2.4] just means that the formula only depends on the K_1-class represented by the elliptic operator A in $K_1(\mathfrak{A}/\mathfrak{K})$. This calculation can be considered to be the heart of the proof of the formula. It is clear that the formula is zero for multiplication operators, i.e. elements in the image of $K_1(C(X))$. Because of Theorem 2 (compare also the proof above), it suffices to check Formula (4) for operators A supported in the interior. But for those, the formula reduces to the classical Atiyah-Singer index formula, which it is true. Again, one can use the double to get exactly the situation of Atiyah-Singer.

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