C*-STRUCTURE AND K-THEORY OF BOUTET DE MONVEL'S ALGEBRA

S.T.MELO, R.NEST, AND E.SCHROHE

Abstract. We consider the norm closure $\mathfrak{A}$ of the algebra of all operators of order and class zero in Boutet de Monvel's calculus on a manifold X with boundary ∂X. We first describe the image and the kernel of the continuous extension of the boundary principal symbol homomorphism to $\mathfrak{A}$. If X is connected and ∂X is not empty, we then show that the K-groups of $\mathfrak{A}$ are topologically determined. In case the manifold, its boundary, and the cotangent space of its interior have torsion free K-theory, we get $K_i(\mathfrak{A}/\mathfrak{K}) \simeq K_i(C(X)) \oplus K_{1-i}(C_0(T^*\dot{X}))$, $i = 0, 1$, with $\mathfrak{K}$ denoting the compact ideal, and $T^*\dot{X}$ denoting the cotangent bundle of the interior. Using Boutet de Monvel's index theorem, we also prove that the above formula holds for $i = 1$ even without this torsion-free hypothesis. For the case of orientable, two-dimensional X, $K_0(\mathfrak{A}) \simeq \mathbb{Z}^{2g+m}$ and $K_1(\mathfrak{A}) \simeq \mathbb{Z}^{2g+m-1}$, where g is the genus of X and m is the number of connected components of ∂X. We also obtain a composition sequence $0 \subset \mathfrak{K} \subset \mathfrak{G} \subset \mathfrak{A}$, with $\mathfrak{A}/\mathfrak{G}$ commutative and $\mathfrak{G}/\mathfrak{K}$ isomorphic to the algebra of all continuous functions on the cosphere bundle of ∂X with values in compact operators on $L^2(\overline{\mathbb{R}}_+)$.

1. Introduction

1.1. **Boutet de Monvel's algebra.** Let X denote an n-dimensional compact manifold, with boundary ∂X, embedded in a closed manifold Ω of the same dimension. Given a pseudodifferential operator P on Ω and u in $C^\infty(X)$, we denote by P_+u the restriction to the interior of X, $\dot{X}$, of P applied to the extension by zero of u to Ω. This gives a continuous mapping $P_+ : C^\infty(X) \to C^\infty(\dot{X})$, completely determined by the restriction of P to $\dot{X}$. It is said that P has the *transmission property* if the image of P_+ is contained in $C^\infty(X)$.

Boutet de Monvel characterized [3] the transmission property for *classical* (i.e., with polyhomogeneous symbols) pseudodifferential operators in terms of certain symmetry conditions satisfied at the boundary by the homogeneuos components of the symbol and their derivatives. In [4], he constructed an algebra containing all classical boundary-value problems on X, together with their parametrices, in the elliptic case. The elements of his calculus, called *Green operators*, are matrices of the form

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X) \\ \oplus \\ C^\infty(\partial X) \end{array} \longrightarrow \begin{array}{c} C^\infty(X) \\ \oplus \\ C^\infty(\partial X) \end{array}, \tag{1}$$

where P is a pseudodifferential operator with the transmission property on X and S is a pseudodifferential operator on the closed manifold ∂X. The operators G, K, and T are regularizing in the interior of X and locally at the boundary are given as pseudodifferential operators on $\mathbb{R}^{n-1}$ with symbols taking values in operators from $\mathcal{S}(\overline{\mathbb{R}}_+)$ to $\mathcal{S}(\overline{\mathbb{R}}_+)$, from $\mathbb{C}$ to $\mathcal{S}(\overline{\mathbb{R}}_+)$, and from $\mathcal{S}(\overline{\mathbb{R}}_+)$ to $\mathbb{C}$, respectively, where $\mathcal{S}(\overline{\mathbb{R}}_+)$ denotes the restriction to the non-negative half-axis $\overline{\mathbb{R}}_+$ of functions in the Schwartz class $\mathcal{S}(\mathbb{R})$. They are called, respectively, singular Green operators, Poisson operators and trace operators.

Being pseudodifferential operators with operator-valued symbols, Green operators have an order assigned to them. Moreover, they also have a *class* (or *type*), related to the order of the derivatives appearing in the boundary condition. There exist order and class reducing operators, which are isomorphisms between suitable Sobolev spaces. For many purposes it is therefore enough to consider the operators of order and class zero. Detailed expositions of Boutet de Monvel's calculus can be found in [18, 35, 39]. The precise estimates satisfied by the symbols defining the operators G, K and T are listed in [18], Definition 2.3.13. One peculiar aspect worth to be mentioned is that, in the polyhomogeneous case, and for operators of order d, the degree of homogeneity of the leading term in the asymptotic expansion for the symbols of G and K is $(d-1)$; while that for P, T and S is, as expected, d.

The set $\mathcal{A}$ of all classical, or polyhomogeneous, Green operators of order zero and class zero on X is an adjoint-invariant sub-algebra of $\mathfrak{L}(\mathfrak{H})$, the bounded operators on the Hilbert space $\mathfrak{H} = L^2(X) \oplus H^{-\frac{1}{2}}(\partial X)$, $H^{-\frac{1}{2}}$ denoting the usual Sobolev space. It is, in fact, a Ψ^*-algebra, in the sense of Gramsch [17] (by [38], Corollary 4.11; the spectral invariance of $\mathcal{A}$ had been shown earlier by Schulze [41]). It then follows that $\mathcal{A}$ is invariant under the holomorphic functional calculus and its K-theory coincides with that of its norm closure, which we denote by $\mathfrak{A}$.

There is some degree of arbitrariness with respect to how one defines the order of Poisson and trace operators. We adopt Grubb's definitions in order to be able to freely quote from her book. Besides, that coincides with Boutet de Monvel's original definitions. On the other hand, since we are interested only in L^2-theory, it would also be natural to say that a trace or a Poisson operator is of order zero if its operator-valued symbol satisfies zero-order norm estimates. If we did that, not only we would get rid of some uncomfortable $\pm 1/2$'s in the exponent ([18], (1.2.19) and (2.3.47), for example), but also the zero-order Green operators would form an algebra of bounded operators on the more familiar Hilbert space $L^2(X) \oplus L^2(\partial X)$. The two approaches are, of course, equivalent. One could go from one algebra to the other by

$$\mathcal{A} \ni \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-\frac{1}{2}} \end{pmatrix},$$

where $\Lambda = (1 - \triangle_{\partial X})^{-\frac{1}{2}}$, $\triangle_{\partial X}$ denoting a second-order nonpositive elliptic operator on ∂X.

1.2. **The boundary principal symbol.** Two homomorphisms are defined on the algebra of all classical Green operators on X ([4], §4): the principal symbol, which we denote by σ; and the boundary principal symbol, which we denote by γ. Given A as in (1), $\sigma(A) = \sigma(P)$ is simply the usual principal symbol of P, regarded as a function on the cosphere bundle S^*X, which we consider for convenience as the set of all unit covectors with respect to a chosen Riemannian metric on X. Since the singular Green operators are regularizing in the interior of X, $\sigma(P) = \sigma(P')$ when $P_+ + G = P'_+ + G'$.

The boundary principal symbol of A, $\gamma(A)$, depends smoothly on covectors in $S^*\partial X$, takes values in Green operators on $\overline{\mathbb{R}}_+$, and needs local coordinates to be described. Let $p(x', x_n, \xi', \xi_n)$, $g(x', \xi', \xi_n, \eta_n)$, $k(x', \xi', \xi_n)$, $t(x', \xi', \xi_n)$, and $s(x', \xi')$ be the symbols of P, G, K, T, and S with respect to coordinates $x = (x', x_n)$ for which the boundary is $\{x_n = 0\}$. Let p_0, g_0, k_0, t_0, and s_0 denote the leading terms in the asymptotics expansions of p, g, k, t and s. For each (x', ξ'), one then defines $\gamma(A)(x', \xi')$ as the Green operator on $\overline{\mathbb{R}}_+$ obtained from the symbols (regarded as functions of ξ_n and η_n) $p_0(x', 0, \xi', \xi_n)$, $g_0(x', \xi', \xi_n, \eta_n)$, $k_0(x', \xi', \xi_n)$, $t_0(x', \xi', \xi_n)$, and $s_0(x', \xi')$. Using the notation of [18], we may write:

$$\gamma(A)(x', \xi') = \begin{pmatrix} p_0(x', 0, \xi', D_n)_+ + g_0(x', \xi', D_n) & k_0(x', \xi', D_n) \\ t_0(x', \xi', D_n) & s_0(x', \xi') \end{pmatrix} \tag{2}$$

For the invariance of this definition, see [35], 2.3.3.1, Theorem 3; and [18], Theorem 2.4.11. Actually, the homomorphism γ depends on the choice of a normal coordinate, even though the algebra and the principal symbol do not. So, let us assume that such a choice has been made; or equivalently, let us suppose that all changes among the above described coordinates preserve x_n.

On $\mathcal{A}$, σ and γ are *-homomorphisms. Moreover, since the Green operators of order and class zero on $\overline{\mathbb{R}}_+$ are bounded, one has $\gamma(A) \in C^\infty(S^*\partial X, \mathfrak{L}(L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C}))$, for all $A \in \mathcal{A}$.

Gohberg [16] and Seeley [43] established the equality between the norm, modulo compacts, of a singular integral operator on a compact manifold and the supremum norm of its symbol. Proofs of that estimate for pseudodifferential operators appeared in [21, 25]. We need the following generalization:

$$\inf_{C \in \mathfrak{K}} ||A + C|| = \max\{||\sigma(A)||, ||\gamma(A)||\}, \text{ for all } A \in \mathcal{A}, \tag{3}$$

with $\mathfrak{K}$ denoting the ideal of the compact operators on $\mathfrak{H}$, $||\sigma(A)||$ the supremum norm of $\sigma(A)$ on S^*X, and $||\gamma(A)||$ the supremum over all (x', ξ') in $S^*\partial X$ of $||\gamma(A)(x', \xi')||_{\mathfrak{L}(L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C})}$. This result can be found in Rempel and Schulze's book ([35], 2.3.4.4, Theorem 1); they credit Grubb and Geymonat [19] for earlier work. Theorem 3.23 in [36] further generalizes (3) to a larger algebra, on which a third symbol is defined.

One obtains from (3), in particular, that σ and γ can be extended to C*-algebra homomorphisms, $\bar\sigma : \mathfrak{A} \to C(S^*X)$ and $\bar\gamma : \mathfrak{A} \to C(S^*\partial X) \otimes \mathfrak{L}(L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C})$, with

$$\inf_{C \in \mathfrak{K}} ||A + C|| = \max\{||\bar\sigma(A)||, ||\bar\gamma(A)||\}, \text{ for all } A \in \mathfrak{A}. \tag{4}$$

We have written $\otimes$ for the C*-algebra tensor product, noting that $C(S^*\partial X)$ is nuclear. Equivalently, for each $A \in \mathfrak{A}$, $\bar\gamma(A)$ is a continuous function on the cosphere bundle of the boundary, with values in bounded operators on $L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$.

1.3. **Statement of results and definitions.** Let $\mathcal{I}$ denote the algebra of all Green operators

$$\begin{pmatrix} \varphi P \psi + G & K \\ T & S \end{pmatrix}, \tag{5}$$

where P is a zero-order classical pseudodifferential operator on X; G, K, T and S have negative order; and φ and ψ belong to $C_c^\infty(\dot X)$, the space of smooth functions with support contained in $\dot X$ (we denote by the same symbols also the operators of multiplication by φ or ψ). In Section 2, we prove that the kernel of $\bar\gamma$ is equal to the norm closure of $\mathcal{I}$, which we denote by $\mathfrak{I}$. The crucial step for that is a norm estimate, modulo $\mathfrak{I}$, stated in Lemma 2; which is, in its essence, a result for manifolds *with* boundary, in the sense that it gives trivial information when applied for a manifold without boundary (for then $\mathfrak{I} = \mathfrak{A}$). The usefulness of this description of $\ker \bar{\gamma}$ to our K-theoretic calculations follows from the fact that the quotient $\mathfrak{I}/\mathfrak{K}$ is isomorphic to the algebra $C_0(S^*\dot{X})$ of all continuous functions on the cosphere bundle of the interior, $S^*\dot{X}$, that vanish at the boundary.

This characterization of $\ker \bar{\gamma}$ is equivalent to saying that, if one enlarges the ideal on the left-hand side of (3), then the principal symbol is not needed on the right. More precisely, we have (Corollary 3):

$$\inf_{A' \in \mathfrak{I}} ||A + A'|| = ||\gamma(A)||, \text{ for all } A \in \mathcal{A}.$$

Let $\mathbb{S}^1 = \{z \in \mathbb{C};\, |z| = 1\}$ and let $U : L^2(\mathbb{S}^1) \to L^2(\mathbb{R})$ denote the unitary mapping

$$Ug(t) = \frac{\sqrt{2}}{1+it}\, g\left(\frac{1-it}{1+it}\right). \tag{6}$$

By $\mathcal{H}_{-1}$ denote the image of $C^\infty(\mathbb{S}^1)$ under U, and by $\mathcal{H}_0 = \mathcal{H}_{-1} \oplus \mathbb{C}$ the direct sum of $\mathcal{H}_{-1}$ with the constant functions. If $p(x', x_n, \xi', \xi_n)$ is the symbol of a classical zero-order pseudodifferential operator with the transmission property on X, with respect to local coordinates as above, then, for each (x', ξ'), $p(x', 0, \xi', \cdot)$ belongs to $\mathcal{H}_0$. Moreover, the image by U of the Hardy space H^2 is equal to $F(L^2(\overline{\mathbb{R}}_+))$, with F denoting the Fourier transform on $\mathbb{R}$; and the bounded operator $p(x', 0, \xi', D_n)_+$ equals $F^{-1}UT_{p_{x',\xi'}}U^{-1}F$, with $T_{p_{x',\xi'}}$ denoting the Toeplitz operator of symbol $p_{x',\xi'}(z) = p(x', 0, \xi', \frac{iz-i}{z+1})$ (we refer to [11] for the definitions of Hardy space and of Toeplitz operators). These are (rephrased) fundamental results for Boutet de Monvel's calculus, their proofs can be found in [4], §1; [35], 2.1; and [18], 2.2. Our definition here of $\mathcal{H}_{-1}$ and $\mathcal{H}_0$ agrees with that of [18], but not with those of [4, 35], where they are denoted by $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

Let $\mathfrak{T}$ denote the C*-algebra of bounded operators on $L^2(\overline{\mathbb{R}}_+)$ generated by $\{p(D)_+;\, p \in \mathcal{H}_0\}$. The above observations prove that $\mathfrak{T}$ is unitarily equivalent to the C*-algebra generated by all Toeplitz operators of continuous symbol; in particular, $\mathfrak{T}$ contains all compact operators on $L^2(\overline{\mathbb{R}}_+)$ ([11], Proposition 7.12). For each (x', ξ'), the entry $g_0(x', \xi', D_n)$ in (2) is an integral operator on $\overline{\mathbb{R}}_+$ with kernel in the space of restrictions to $(\overline{\mathbb{R}}_+)^2$ of functions in the Schwartz space on $\mathbb{R}^2$ ([18], Section 2.3). In particular, $g_0(x', \xi', D_n)$ is compact, and the upper left corner of the matrix in (2) belongs to $\mathfrak{T}$ for every (x', ξ'), not depending on the way we write $P_+ + G$ as the sum of a truncated pseudodifferential operator with the transmission property and a singular Green operator (see the proof of Lemma 1).

The *symbol-kernel* approach of [18], Section 2.3, also makes it easier to give a precise description of the range of the off-diagonal terms of the boundary principal symbol: for each (x', ξ'), $t_0(x', \xi', D_n) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathbb{C}$ is the operator $\varphi \mapsto \int \tilde{t}_0 \varphi$, where $\tilde{t}_0 \in \mathcal{S}(\overline{\mathbb{R}}_+)$ is the Fourier transform of $t_0(x', \xi', \cdot)$ divided by 2π; and $k_0(x', \xi', D_n) : \mathbb{C} \to \mathcal{S}(\overline{\mathbb{R}}_+)$ is the operator of multiplication by $\tilde{k}_0 \in \mathcal{S}(\overline{\mathbb{R}}_+)$, the inverse Fourier transform of $k_0(x', \xi', \cdot)$.

Any $A \in \mathfrak{A}$ can be written as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, with the A_{jk}'s denoting the obvious compositions of A with orthogonal projections of $\mathfrak{H}$ onto its subspaces

$L^2(X)$ and $H^{-\frac{1}{2}}(\partial X)$. By definition, $\mathcal{A}$ is invariant under multiplication by these projections. We also denote $\mathfrak{A}_{jk} = \{A_{jk}; A \in \mathfrak{A}\}$. Completing the matrix with zeros, it is clear that $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$ may be regarded as subalgebras of $\mathfrak{A}$, with non-unital inclusions; while $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$ are only subspaces. We will use the same notation for any subalgebra of $\mathfrak{A}$ and a similar notation for homomorphisms.

The preceding discussion shows that $\bar{\gamma}$ maps $\mathfrak{A}$ to

$$C(S^*\partial X) \otimes \begin{pmatrix} \mathfrak{T} & L^2(\overline{\mathbb{R}}_+) \\ L^2(\overline{\mathbb{R}}_+)^* & \mathbb{C} \end{pmatrix}. \tag{7}$$

In Section 3, we prove that the image of $\bar{\gamma}$, $\operatorname{Im}\bar{\gamma}$, is yet a proper subalgebra of the algebra in (7). The non-surjectiveness is observed only at the upper-left corner, as explained in the next two paragraphs. First, let us notice that every $p \in \mathcal{H}_{-1}$ vanishes at infinity (that follows immediately from the definitions of U and of $\mathcal{H}_{-1}$). Hence, $p(\infty)$ is defined for every p in $\mathcal{H}_0$.

Let $\mathfrak{K}$ also denote the ideal of compact operators on $L^2(\overline{\mathbb{R}}_+)$ (we will denote by the same letter the compact ideal on any of the Hilbert spaces of this paper; except in the few cases when a distinction between some of them will be needed). It is a result of Coburn [6, 7] and Douglas [10] that $\mathfrak{T}/\mathfrak{K}$ is isomorphic to $C(\mathbb{R}\cup\{\infty\}) \simeq C(\mathbb{S}^1)$, with isomorphism induced by the symbol mapping (see [11]; Theorem 7.23, and the bibliographical notes of Chapter 7, where the influence of previous work of Gohberg is acknowledged). The mapping $p(D)_+ \mapsto p(\infty)$ extends therefore to a *-homomorphism $\lambda : \mathfrak{T} \to \mathbb{C}$ containing $\mathfrak{K}$ in its kernel, which we denote $\mathfrak{T}_0$.

We show that $\operatorname{Im}\bar{\gamma}_{11}$ contains $C(S^*\partial X) \otimes \mathfrak{T}_0$, contains also $C(\partial X) \otimes \mathbb{C}$, and that's all. The direct sum of these two algebras gives $\operatorname{Im}\bar{\gamma}_{11}$; while $\bar{\gamma}_{jk}$ is surjective if $j \neq 1$ or $k \neq 1$. This description of $\operatorname{Im}\bar{\gamma}$ is stated in Theorem 2, in a form more suitable for applications to K-theory. It is precisely the fact that all $\mathfrak{T}_0$-valued functions, but not all the $\mathfrak{T}$-valued ones, are contained in $\operatorname{Im}\bar{\gamma}_{11}$ that allows our very explicit computation of K-groups: since $K_*(\mathfrak{T}_0) = 0$, it follows from Theorem 2 that $\operatorname{Im}\bar{\gamma}$ and $C(\partial X)$ have isomorphic K-theory (Corollary 8).

In Section 4, using that there exists a nonvanishing section of the cotangent bundle (Proposition 9; for that, it is required that X is connected and that ∂X is nonempty), we reduce to a purely topological problem the analysis of the index and exponential mappings in the six-term cyclic exact sequence associated to

$$0 \to \mathfrak{I}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{I} \to 0. \tag{8}$$

Theorem 3 then solves the problem of computing the K-theory of $\mathfrak{A}/\mathfrak{K}$, in the sense that both $K_0(\mathfrak{A}/\mathfrak{K})$ and $K_1(\mathfrak{A}/\mathfrak{K})$ are put in the middle of short exact sequences of abelian groups determined by the topologies of the manifold and of the cotangent bundle of the interior, $T^*\dot{X}$. In case X, ∂X, and $T^*\dot{X}$ have torsion-free K-theory, we get (Corollary 12):

$$K_i(\mathfrak{A}/\mathfrak{K}) \simeq K_i(C(X)) \oplus K_{1-i}(C_0(T^*\dot{X})),\ i = 0, 1, \tag{9}$$

where $C_0(T^*\dot{X})$ denotes the set of all functions on $T^*\dot{X}$ which get arbitrarily small outside compacts.

In Propositions 13 and 14, we prove that the three groups $K_0(\mathfrak{A}/\mathfrak{K})$, $K_0(\mathfrak{A})$, and $K_0(\mathcal{A})$ are isomorphic; while $K_1(\mathfrak{A})$ and $K_1(\mathcal{A})$ are isomorphic to the kernel of the Fredholm index mapping $K_1(\mathfrak{A}/\mathfrak{K}) \to \mathbb{Z}$, which is surjective.

Proposition 11 gives an isomorphism between $K_i(C_0(S^*\dot{X}))$ and the direct sum of $K_i(C_0(\dot{X}))$ and $K_{1-i}(C_0(T^*\dot{X}))$, $i = 0, 1$. Moreover, with respect to this isomorphism, the canonical projection of $K_i(C_0(S^*\dot{X}))$ onto $K_{1-i}(C_0(T^*\dot{X}))$ is equivalent to the index mapping for the exact sequence $0 \to C_0(T^*\dot{X}) \to C_0(B^*\dot{X}) \to C_0(S^*\dot{X}) \to 0$, with $B^*\dot{X}$ denoting the bundle of unit balls over $\dot{X}$, which may also be regarded as the radial compactfication of T^*X.

In Section 5, we prove (9) for $i = 1$ without the torsion-free assumptions of Corollary 12. The reason why we get for K_1 a better answer than for K_0 is the fact, proven by Boutet de Monvel [4], that the classical *difference bundle*, defined by the principal symbol, induces a homomorphism from $K_1(\mathfrak{A}/\mathfrak{K})$ to $K_0(C_0(T^*\dot{X}))$, which we will denote by `ind`. Moreover, since the difference bundle construction is precisely the index mapping for topological K-theory (Proposition 15), the composition of `ind` with the homomorphism i_* induced by the inclusion of $\mathfrak{I}/\mathfrak{K}$ into $\mathfrak{A}/\mathfrak{K}$ is equivalent to the canonical projection $K_1(C_0(\dot{X})) \oplus K_0(C_0(T^*\dot{X})) \to K_0(C_0(T^*\dot{X}))$. Even though i_* is not necessarily injective (for the example considered in Section 6, $\ker i_* \simeq \mathbb{Z}^{m-1}$, $m \geq 1$), its restriction to $K_0(C_0(T^*\dot{X}))$ is. That is how $K_0(C_0(T^*\dot{X}))$ injects into $K_1(\mathfrak{A}/\mathfrak{K})$. The injection of $K_1(C(X))$ is induced by the embedding of $C(X)$ as multiplication operators. All that is summarized in Theorem 4.

With respect to the isomorphism $K_1(\mathfrak{A}/\mathfrak{K}) \simeq K_1(C(X)) \oplus K_0(C_0(T^*\dot{X}))$ of Theorem 4, we show that `ind` corresponds to the canonical projection onto $K_0(C_0(T^*\dot{X}))$ (Corollary 18). Boutet de Monvel's index theorem ([4], Theorem 5.22) then implies that $K_1(\mathfrak{A})$ is isomorphic to $K_1(C(X)) \oplus \ker \chi$, where $\chi : K_0(C_0(T^*\dot{X})) \to \mathbb{Z}$ denotes the topological index.

In Section 6, we assume that X is a connected two-dimensional orientable manifold with nonempty boundary. We denote its genus by g and by m the number of connected components of the boundary. We then apply the results of Section 4 to prove that $K_0(\mathfrak{A}/\mathfrak{K})$ and $K_1(\mathfrak{A}/\mathfrak{K})$ are both isomorphic to $\mathbb{Z}^{2g+m}$. The standard description of a closed surface as a polygon with sides identified is used to compute the K-theory of X and of $\dot{X}$.

Under our initial assumption that X is an arbitrary compact manifold with boundary, in Section 7 we give the following *composition sequence* (in the sense of [9], 4.3.2) for $\mathfrak{A}$:

$$0 \subset \mathfrak{K} \subset \mathfrak{G} \subset \mathfrak{A}, \tag{10}$$

where $\mathfrak{G}$ denotes the closure of $\mathcal{G}$, the algebra of all Green operators A as in (1) with P of negative order. All commutators of $\mathcal{A}$ belong to $\mathcal{G}$, as follows from the rules of Boutet de Monvel's calculus ([18], Section 2.6, for example). In Theorem 5, we prove that the principal symbol induces an isomorphism between $\mathfrak{A}/\mathfrak{G}$ and $C(S^*X/\sim)$, where $S^*X/\sim$ denotes the quotient of S^*X by the equivalence relation that identifies the *North and South poles* over each point in ∂X (i.e., the two covectors that vanish on vectors tangent to the boundary). Moreover, it follows immediately from (3) that the boundary principal symbol induces an isometry on $\mathfrak{G}/\mathfrak{K}$. Then it is not hard to see (Theorem 6) that the image of that isometry is equal to $C(S^*\partial X) \otimes \mathfrak{K}_{\overline{\mathbb{R}}_+}$, with $\mathfrak{K}_{\overline{\mathbb{R}}_+}$ denoting the ideal of compact operators on $L^2(\overline{\mathbb{R}}_+)$.

A natural problem posed by this composition sequence is to understand the connecting mappings $K_i(C(S^*X/\sim)) \to K_{1-i}(C(S^*\partial X))$, $i = 0, 1$ in the six-term exact sequence associated to $0 \to \mathfrak{G}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{G} \to 0$. That could lead to another way of computing the K-groups of $\mathfrak{A}$. Moreover, as already suggested by the case of orientable surfaces, there may exist interesting connections between those mappings and the topology of the manifold.

1.4. **Related results.** There are many other similar composition sequences for C*-algebras generated by pseudodifferential operators. Closer to us, Cordes [8] used C*-algebra techniques to solve boundary value problems on the half space. His algebra $\mathfrak{A}_C$ has a composition sequence $0 \subset \mathfrak{K} \subset \mathfrak{E} \subset \mathfrak{A}_C$; with the cosphere bundle contained in the Gelfand space of the commutative C*-algebra $\mathfrak{A}_C/\mathfrak{E}$, and $\mathfrak{E}/\mathfrak{K}$ isomorphic to the algebra of $\overline{\mathbb{R}}_+$-compact-operator-valued continuous functions on a closed subset of a compactification of the cotangent bundle of the boundary.

For the C*-algebra $\mathfrak{A}_M$ generated by Melrose's b-pseudodifferential operators on a compact manifold with boundary, Lauter [26] found the compostion sequence $0 \subset \mathfrak{K} \subset \mathfrak{E} \subset \mathfrak{A}_M$, with $\mathfrak{A}_M/\mathfrak{E}$ isomorphic (via the principal symbol) to the algebra of continuous functions on the cosphere bundle, and $\mathfrak{E}/\mathfrak{K}$ isomorphic to a direct sum (indexed by the connected components of the boundary) of algebras of continuous ∂X-compact-operator-valued functions on $\mathbb{R}$.

For the case of a manifold with corners of dimension n, Melrose and Nistor [29] obtained a composition sequence $0 \subset \mathfrak{K} = \mathfrak{I}_n \subset \cdots \subset \mathfrak{I}_1 \subset \mathfrak{I}_0 \subset \mathfrak{A}_M$, with $\mathfrak{A}_M/\mathfrak{I}_0$ isomorphic to the continuous functions on the cosphere bundle, and each $\mathfrak{I}_l/\mathfrak{I}_{l+1}$ isomorphic to direct sums of algebras of compact-operator-valued functions on $\mathbb{R}$. They explicitly computed the connecting mappings for the six-term exact sequences associated to each of the quotients $\mathfrak{I}_l/\mathfrak{I}_{l+1}$. Their results were generalized for algebras of pseudodifferential operators on groupoids by Monthubert [30, 31]. Also in the language of groupoids, the K-theory of $\mathfrak{A}_M$ was computed by Lauter, Monthubert and Nistor [27], for the case of a manifold with connected boundary (no boundary faces of codimension larger than one).

Index theorems for operators in Boutet de Monvel's calculus have been established by several authors: Boutet de Monvel [4], Fedosov [13], Rempel and Schulze [35], Grubb [18]. The corresponding formulas, however, are not very explicit. The present project is part of our intention to reconsider the problem under the perspective of noncommutative geometry. Computing the K-theory as well as the Hochschild and cyclic cohomology [34], we hope to be able to adapt the index-theoretic methods developed by Nest and Tsygan [32].

Boutet de Monvel's algebra has also been studied under other operator-algebraic aspects. Fedosov, Golse, Leichtnam and Schrohe [14] showed that there exists a unique continuous trace on it which extends Wodzicki's noncommutative residue [46]. Its relation to Dixmier's trace has been studied in [33]. Finally, it was proven by Grubb and Schrohe [20] that this trace is indeed given as a residue in the meromorphic extension of a suitable operator trace.

1.5. **Vector Bundles.** It is straightforward to generalize the C*-algebra structure results of Sections 2, 3 and 7 for algebras of Green operators acting between sections of vector bundles. The problem of how to define the boundary principal symbol has been addressed in [35], 2.3.3.1, and in [18], page 228. The crucial norm estimate (3) is proven in [35], 2.3.4.4, already in this more general formulation.

For the K-theory applications of Sections 4, 5, and 6, it is enough to consider the case of trivial bundles. Indeed, we are now going to prove that the corresponding algebras for any two choices of vector bundles are strongly Morita equivalent (this was suggested to us by Wodzicki). But then, strongly Morita equivalent C*-algebras have the same K-theory. That follows, for C*-algebras possessing countable approximate identities, from a theorem of Brown, Green and Rieffel [5]. Exel [12] proved the general case, by explicitly constructing the isomorphism predicted by Brown, Green and Rieffel.

Let E be a smooth rank-k vector bundle over X, and F be a rank-l smooth vector bundle over ∂X, $k > 0$ and $l \geq 0$ (the case $E = 0$ belongs to classical index theory on closed manifolds). Let $\mathcal{A}_{EF}$ denote the algebra of all Green operators $A : C^\infty(E) \oplus C^\infty(F) \to C^\infty(E) \oplus C^\infty(F)$, and let $\mathfrak{A}_{EF}$ denote the norm closure of $\mathcal{A}_{EF}$ in $\mathfrak{L}(\mathfrak{H}_{EF})$, the algebra of the bounded operators on $\mathfrak{H}_{EF} = L^2(E) \oplus H^{-\frac{1}{2}}(F)$. We are going to show that $\mathfrak{A}_{EF}$ and $\mathfrak{A}_{kl}$ are strongly Morita equivalent, with $\mathfrak{A}_{kl}$ denoting the closure of the algebra of all Green operators acting between sections of the rank-k trivial bundle over X and the rank-l trivial bundle over ∂X.

Let $\mathcal{X}$ denote the set of all Green operators mapping sections of E and F to sections of the trivial bundles of rank k over X and of rank l over ∂X, and let $\mathfrak{X}$ denote the closure of $\mathcal{X}$ in $\mathfrak{L}(\mathfrak{H}_{EF}, \mathfrak{H}_{kl})$, $\mathfrak{H}_{kl} = L^2(X; \mathbb{C}^k) \oplus H^{-\frac{1}{2}}(\partial X; \mathbb{C}^l)$. We may define operator-valued inner products on $\mathfrak{X}$ by $\langle A, B\rangle_{\mathfrak{A}_{kl}} = AB^*$ and $\langle A, B\rangle_{\mathfrak{A}_{EF}} = A^*B$. Equipped with $\langle\cdot,\cdot\rangle_{\mathfrak{A}_{kl}}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{A}_{EF}}$, it follows immediately from the rules of Boutet de Monvel's calculus that $\mathfrak{X}$ becomes a Hilbert $\mathfrak{A}_{kl}$-$\mathfrak{A}_{EF}$-bimodule. To prove the strong Morita equivalence (as defined in [12], Section 5) of these two algebras, it is enough to show that $\mathfrak{X}$ is simultaneously left and right-full. In other words, we must show that the linear span of all A^*B, with A and B in $\mathfrak{X}$, is dense in $\mathfrak{A}_{EF}$; and that the linear span of all AB^* is dense in $\mathfrak{A}_{kl}$.

Let there be given arbitrary $C \in \mathcal{A}_{EF}$, and φ and $\psi \in C^\infty(X)$ such that their supports are contained in a trivial open set for E and the intersection of their supports with ∂X is contained in a trivial open set for F. To prove the first (the other is analogous) density statement, above, it clearly suffices to obtain A and B in $\mathcal{X}$ such that $\varphi C\psi = A^*B$ (given $\rho \in C^\infty(X)$, we denote also by ρ the operator $(f, g) \in C^\infty(E) \oplus C^\infty(F) \mapsto (\rho f, (\rho|_{\partial X})g))$. This can be done by letting B have the same local expression as $\varphi C\psi$ and letting A be locally given by $\tilde{\psi}$, for some $\tilde{\psi} \in C^\infty(X)$ with support slightly larger than suppψ and equal to one on a neighborhood of it. That shows that $\mathfrak{A}_{EF}$ and $\mathfrak{A}_{kl}$ are strongly Morita equivalent.

Finally, it is straightforward to prove that $\mathfrak{A}_{kl}$ and $\mathfrak{A}$ are strongly Morita equivalent, taking for the Hilbert bimodule the closure of set of all Green operators from $C^\infty(X, \mathbb{C}^k) \oplus C^\infty(\partial X, \mathbb{C}^l)$ to $C^\infty(X) \oplus C^\infty(\partial X)$. One nontrivial step, needed only to deal with the case $l = 0$, is to prove that every S belongs to the linear span of all TK, for T, K and S as in (1).

2. The kernel of $\bar{\gamma}$

Let us first note that $\mathcal{I}$ is contained in the kernel of γ, as implied by the facts that G, K, T and S in (5) are of negative order, and that the principal symbol of $\varphi P\psi$ vanishes over the boundary.

Another easy observation is that $\mathfrak{I}$ contains the compact ideal $\mathfrak{K}$. That follows because the set of all operators with smooth kernel is dense in $\mathfrak{K}$ and any such operator can be written as in (5), with $P = 0$ and G, K, T and S of order $-\infty$.

Lemma 1. *Every $A \in \mathcal{A}$ such that $\gamma(A) = 0$ belongs to $\mathfrak{I}$.*

Proof: Let $A \in \mathcal{A}$ be given, with $\gamma(A) = 0$. It follows from the definition that $A_{jk} \in \mathcal{I}_{jk}$, if $j \neq 1$ or $k \neq 1$. If $A_{11} = P_+ + G$, then $\gamma(P_+)(x', \xi') = -\gamma(G)(x', \xi')$ for every $(x', \xi') \in S^*\partial X$. In Section 1, we remarked that $\gamma(G)(x', \xi') \in \mathfrak{K}$ for every $(x', \xi') \in S^*\partial X$ and that $p(D)_+ \mapsto p$ defines an isomorphism of $\mathfrak{T}/\mathfrak{K}$ with $C(\mathbb{R} \cup \{\infty\})$. From that it follows that the principal symbol of P, $p_0 \in C^\infty(S^*X)$, vanishes over the boundary ∂X and, hence, that $\gamma(P_+)$ and $\gamma(G)$ both vanish. Then, by (3), G is compact ($\sigma(G) = 0$, by definition) and it suffices to show that P_+ is in $\mathfrak{I}_{11}$.

Let $\rho \in C^\infty(X)$ be a *boundary defining function*, i.e., ρ is positive on $\dot{X}$ and vanishes with nonzero derivative on ∂X. Let Q be a zero-order classical pseudodifferential operator on X with principal symbol $q \in C^\infty(S^*X)$ defined by $p_0 = \rho q$. We have $P_+ \equiv \rho Q_+$, modulo a compact operator, since pseudodifferential operators of negative order on compact manifolds are compact. Moreover, Q has the transmission property, but we will not need that. Now let $\varphi_k \in C_c^\infty(\dot{X})$ be equal to one on some sequence of compacts exhausting $\dot{X}$. Since $\varphi_k \rho$ converges to ρ uniformly on X, we have that $\varphi_k \rho Q_+$ converges to ρQ_+ in $\mathfrak{L}(L^2(X))$.

So, it suffices to prove that $\varphi Q_+ \in \mathfrak{I}_{11}$ if φ belongs to $C_c^\infty(\dot{X})$. Let $\psi \in C_c^\infty(\dot{X})$ be equal to one in a neighborhood of φ. Then we have: $\varphi Q_+ = \psi[\varphi, Q_+] + \psi Q_+ \varphi$. The commutator $[\varphi, Q_+]$ is compact, since it is equal to $(\varphi Q - Q\varphi)_+$ and $(\varphi Q - Q\varphi)$ has negative order. That proves the lemma, since $\psi Q_+ \varphi = \psi Q \varphi$. $\square$

Thus, we have $\mathcal{I} \subset \ker\gamma \subset \mathfrak{I}$, and, hence, $\overline{\ker\gamma} = \mathfrak{I}$. It is obvious that $\overline{\ker\gamma} \subseteq \ker\bar{\gamma}$. We prove next that $\ker\bar{\gamma} \subseteq \mathfrak{I}$. Let $A \in \ker\bar{\gamma}$ be given. If $j \neq 1$ or $k \neq 1$, we then have:

(11)
$$\inf_{A' \in \mathfrak{I}} ||A_{jk} + A'_{jk}|| \leq \inf_{C \in \mathfrak{K}} ||A_{jk} + C|| = \max\{||\bar{\sigma}_{jk}(A)||, ||\bar{\gamma}_{jk}(A)||\} = ||\bar{\gamma}_{jk}(A)||,$$

since, by definition, $\bar{\sigma}_{jk} = 0$ unless $j = k = 1$. We have used (4) for the matrix that has A_{jk} in its (j,k)-entry and zero in the others. It follows from (11) that $\ker\bar{\gamma}_{jk} \subseteq \mathfrak{I}_{jk}$ if $j \neq 1$ or $k \neq 1$.

To show that $\ker\bar{\gamma}_{11} \subseteq \mathfrak{I}_{11}$ (and hence prove that $\ker\bar{\gamma} = \mathfrak{I}$), it is enough to prove Lemma 2, below. Indeed, the left-hand side of (13) is by definition greater than or equal to $\inf\{||P_+ + G + A'_{11}||; A' \in \mathfrak{I}\}$. Moreover, since both sides of the inequality depend continuously on $A = P_+ + G$, proving (13) will imply that it holds also, for the extension $\bar{\gamma}$, if $P_+ + G$ is replaced by a general $A \in \mathfrak{A}_{11}$.

For each $s > 0$, let us define the unitary operator κ_s on $L^2(\overline{\mathbb{R}}_+)$ by $(\kappa_s\psi)(t) = \sqrt{s}\psi(st)$. It is straightforward to check that

$$\kappa_s^{-1} b(D)_+ \kappa_s = b(sD)_+, \text{ for all } s > 0, \tag{12}$$

with $b(D)_+$ denoting the operator on $L^2(\overline{\mathbb{R}}_+)$ obtained by truncating the Fourier multiplier $b(D)$, for any measurable function b on $\mathbb{R}$.

Lemma 2. *There exists a positive constant c, determined only by X, such that, for every pseudodifferential operator with the transmission property on X, and for every singular Green operator G, both polyhomogeneous and of order zero, we have:*

$$\inf_{\varphi,\psi,Q} ||P_+ + G + \varphi Q \psi|| \leq c||\gamma(P_+ + G)||, \tag{13}$$

where the infimum is taken over all φ and ψ in $C_c^\infty(\dot{X})$ and all zero-order classical pseudodifferential operators Q on X.

Proof: The closure of set of all such $\varphi Q\psi$ contains the compact operators (it is enough to take Q's of order $-\infty$). Moreover, (3) applied to the matrix that has G in the upper left corner and zero in the other entries gives $\inf_{C\in\mathfrak{K}}||G+C|| = ||\gamma(G)||$. This implies (13), with $c=1$, for the case $P_+ = 0$.

Next we prove (13) for the case $G=0$. We have already noticed that the left-hand side of (13) is not smaller than the infimum of $\{||P_+ + G + A'_{11}||; A' \in \mathfrak{I}\}$. They are, in fact, equal, since any singular Green operator of negative order on X is compact, and the compacts are contained in the closure of the set of all $\varphi Q\psi$. Let $\{\varphi_1, \cdots, \varphi_d\}$ be a partition of unity on X such that, whenever the supports of φ_i and φ_j intersect, their union is contained in the domain of a chart $\chi_{ij} : U_{ij} \to \tilde{U}_{ij}$. If supp$\varphi_i$ and suppφ_j do not intersect, then $\varphi_i P_+ \varphi_j$ is regularizing, hence compact. Since $\mathfrak{K} \subseteq \mathfrak{I}_{11}$, we get:

$$\inf_{A'\in\mathfrak{I}_{11}} ||P_+ + A'|| \leq \sum_{\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \emptyset} \inf_{A'\in\mathfrak{I}_{11}} ||\varphi_i P_+ \varphi_j + A'||. \tag{14}$$

If U_{ij} does not intersect the boundary, then $\varphi_i P_+ \varphi_j = \varphi_i P \varphi_j \in \mathcal{I}_{11}$. For each (i,j) in the sum, above, we may therefore suppose that U_{ij} intersects the boundary, given by $x_n = 0$, and denote by $p(x,\xi)$ the local symbol of P for that chart. Let P_{ij} denote the pullback by χ_{ij} of the pseudodifferential operator $\tilde{P}_{ij}$ on $\bar{\mathbb{R}}^n_+$ of amplitude

$$q_{ij}(x,y,\xi) = p(x',0,\xi)\varphi_i(\chi_{ij}^{-1}(x))\varphi_j(\chi_{ij}^{-1}(y)). \tag{15}$$

P_{ij} is a classical pseudodifferential operator with the transmission property on X, such that $P_{ij+} - \varphi_i P_+ \varphi_j$ is in the kernel of γ. It follows from Lemma 1 that $P_{ij+} - \varphi_i P_+ \varphi_j$ belongs to $\mathfrak{I}_{11}$ and, hence, that

$$\inf_{A'\in\mathfrak{I}_{11}} ||\varphi_i P_+ \varphi_j + A'|| = \inf_{A'\in\mathfrak{I}_{11}} ||P_{ij+} + A'|| \leq \inf_{C\in\mathfrak{K}} ||P_{ij+} + C||. \tag{16}$$

Let us denote by P'_{ij} the pseudodifferential operator on $\mathbb{R}^n$ with amplitude defined by the same formula (15) as $\tilde{P}_{ij}$, simply assuming, as we may, that the φ's and χ's are restrictions of functions and charts on the neighboring manifold Ω. The classical estimate for the norm, modulo compacts, of a pseudodifferential operator in terms of the supremum-norm of its principal symbol ([25], Theorem A.4; or [21], Theorem 3.3) implies the existence of compact operators C'_{ij} on $L^2(\mathbb{R}^n)$, such that $||P'_{ij} + C'_{ij}||$ is bounded by two times the supremum-norm of the principal symbol of P'_{ij}; which is bounded by the supremum on the right-hand side of (17), below, since $|\varphi_k| \leq 1$ for all k. If $\tilde{C}_{ij}$ denotes the compact operator on $L^2(\bar{\mathbb{R}}^n_+)$ obtained from C'_{ij} by truncation, it is obvious that $||\tilde{P}_{ij+} + \tilde{C}_{ij}||$ is bounded by $||P'_{ij} + C'_{ij}||$. This gives:

$$||\tilde{P}_{ij+} + \tilde{C}_{ij}|| \leq 2\sup\{|p_0(x',0,\xi)|;\ (x',0) \in \tilde{U}_{ij},\ \xi \neq 0\}, \tag{17}$$

where $p_0(x,\xi)$ (smooth for $\xi \neq 0$) denotes the zero-order homogeneous principal part of $p(x,\xi)$.

This is the most delicate point of this proof: For each (x',ξ') with $\xi' \neq 0$, $p_0(x',0,\xi',\cdot)$ belongs to $\mathcal{H}_0$, and

$$\sup_{\xi_n\in\mathbb{R}} |p_0(x',0,\xi',\xi_n)| = ||p_0(x',0,\xi',D_n)_+||_{\mathfrak{L}(L^2(\overline{\mathbb{R}}_+))}. \tag{18}$$

That again follows from the isomorphism $\mathfrak{T}/\mathfrak{K} \simeq C(\mathbb{R}\cup\{\infty\})$, or, in a more classical language, from Lemma 3.1.5 of [18]. Since

$$\sup_{\xi'\neq 0}\sup_{\xi_n\in\mathbb{R}} |p_0(x',0,\xi',\xi_n)| = \sup_{\xi\neq 0}|p_0(x',0,\xi)|,$$

the right-hand side of (17) equals $2\sup_{\xi'\neq 0}||p_0(x',0,\xi',D_n)_+||$. It follows from the homogeneity of p_0 and (12) that $p_0(x',0,s\xi',D_n)_+ = \kappa_s p_0(x',0,\xi',D_n)_+\kappa_s^{-1}$, for all $s>0$, and hence $||p_0(x',0,\xi',D_n)_+||$ is independent of $|\xi'|$. We then get:

$$||\tilde{P}_{ij_+} + \tilde{C}_{ij}|| \leq 2||\gamma(P_+)||. \tag{19}$$

Let $C_{ij}\in\mathfrak{K}$ denote the pullback of (a restriction of) $\tilde{C}_{ij}$ by the chart χ_{ij} (C_{ij} vanishes on functions whose support does not intersect the closure of U_{ij}). There is a constant c_1, depending only on our choice of norm on $L^2(X)$, such that

$$||P_{ij_+} + C_{ij}|| \leq c_1||\tilde{P}_{ij_+} + \tilde{C}_{ij}||. \tag{20}$$

Estimates (14), (16), (19) and (20) imply (13), for the case $G=0$, with $c = 2c_1d^2$.

To treat the case when both P_+ and G are nontrivial, we use that $\gamma(G)(x',\xi')$ is compact for each $(x',\xi')\in S^*\partial X$, and again the isomorphism $\mathfrak{T}/\mathfrak{K}\simeq C(\mathbb{S}^1)$, to get:

$$||\gamma(P_+)(x',\xi')|| = \inf_{C\in\mathfrak{K}}||\gamma(P_+)(x',\xi') + C|| \leq ||\gamma(P_+)(x',\xi') + \gamma(G)(x',\xi')||.$$

Taking the supremum on both sides of this inequality, we see that $||\gamma(P_+)|| \leq ||\gamma(P_+ + G)||$, and, hence, $||\gamma(G)|| \leq 2||\gamma(P_+ + G)||$. Since

$$\inf_{\varphi,\psi,Q}||P_+ + G + \varphi Q\psi|| \leq \inf_{\varphi,\psi,Q}||P_+ + \varphi Q\psi|| + \inf_{\varphi,\psi,Q}||G+\varphi Q\psi||,$$

the proof is complete, with $c = 2(1+c_1d^2)$. □

Corollary 3. *$\bar{\gamma}$ induces a C^*-algebra isomorphism between $\mathfrak{A}/\mathfrak{I}$ and $\operatorname{Im}\bar{\gamma}$. Equality, with $c=1$, therefore holds in (13).*

For the sake of this argument, let $\mathfrak{K}_X$ and $\mathfrak{K}_\oplus$ denote the ideals of compact operators on $L^2(X)$ and on $\mathfrak{H}$, respectively. There is an obvious injection of $\mathfrak{I}_{11}/\mathfrak{K}_X$ into $\mathfrak{I}/\mathfrak{K}_\oplus$; which is also surjective because, by definition, all entries of the matrix in (5) are compact, except possibly the upper left one. The statement about $\mathfrak{I}/\mathfrak{K}$ in the theorem, below, follows therefore from the estimate

$$\inf_{C\in\mathfrak{K}}||\varphi P\psi + C|| = \sup_{S^*\dot{X}}|\varphi\psi\sigma(P)|,$$

which again follows from the classical estimate quoted between (16) and (17) (see also [23], Section 2). We have proven:

Theorem 1. *The kernel of $\bar{\gamma}$ is equal to $\mathfrak{I}$. Moreover, $\mathfrak{I}$ contains the compact ideal $\mathfrak{K}$ and $\mathfrak{I}/\mathfrak{K}$ is isomorphic to $C_0(S^*\dot{X})$, with isomorphism induced by the principal symbol.*

3. The image of $\bar{\gamma}$

It is of central importance for the computation of the K-theory of Boutet de Monvel's algebra that the upper left corner of the image of $\bar{\gamma}$, $\operatorname{Im}\bar{\gamma}_{11}$, is isomorphic to the direct sum of $C(\partial X)$ and an ideal of the Toeplitz algebra with vanishing K-theory. This description is the essential result of this section; what we state about $\operatorname{Im}\bar{\gamma}_{jk}$, for $j\neq 1$ or $k\neq 1$, is already contained in [35], 2.3.4.4, Corollary 2.

Every smooth function on $S^*\partial X$ is the principal symbol of a zero-order classical pseudodifferential operator on the closed manifold ∂X. $C^\infty(S^*\partial X)$ is therefore contained in $\operatorname{Im}\gamma_{22}$ and, hence, $\operatorname{Im}\bar{\gamma}_{22} = C(S^*\partial X)$. The fact that $\operatorname{Im}\bar{\gamma}_{21} = C(S^*\partial X) \otimes L^2(\overline{\mathbb{R}}_+)^*$ (so, $\bar{\gamma}_{21}$ is surjective, if we define $\bar{\gamma}$ taking values in the C*-algebra in (7)) follows from the next lemma, by a partition-of-unity argument.

Given $\varphi \in L^2(\overline{\mathbb{R}}_+)$, we will denote by $\langle\varphi|$ the linear functional $\psi \mapsto \int \varphi\psi$; and by $|\varphi\rangle$ the linear map, from $\mathbb{C}$ to $L^2(\overline{\mathbb{R}}_+)$, of multiplication by φ.

Lemma 4. *Let $V \subset \partial X$ be an open set whose closure is contained in the domain of a chart of ∂X. The space $C_0(S^*V, L^2(\overline{\mathbb{R}}_+)^*)$ of all continuous functions, from $\overline{S^*V}$ to $L^2(\overline{\mathbb{R}}_+)^*$, which vanish on every unit covector over the boundary of V is contained in the image of $\bar{\gamma}_{21}$.*

Proof: Let $\chi : U \to \tilde{U} \subseteq \mathbb{R}^{n-1}$ be a chart of ∂X such that $\bar{V} \subset U$. Let there be given p in $C_c^\infty(S^*V)$ (i.e., $p \in C^\infty(S^*\partial X)$ and its support is contained in S^*V) and φ in $C_c^\infty(\mathbb{R}_+)$. Let us denote by $\tilde{p}(x', \xi')$, $(x', \xi') \in \tilde{U} \times \mathbb{R}^{n-1}$, the local expression (smooth for $\xi' \neq 0$) of the zero-degree homogeneous extension of p to the cotangent bundle of ∂X. We now choose an *excision function* ω (i.e., $\omega \in C^\infty(\mathbb{R})$ vanishes on a neighborhood of the origin and $\omega(t) \equiv 1$ for sufficiently large t) and define $t(x', \xi', \xi_n) = 2\pi\omega(|\xi'|)\tilde{p}(x', \xi')\check{\varphi}(\xi_n/|\xi'|)$, with $\check{}$ denoting the inverse Fourier transform, and $|\xi'|$ the euclidean norm of $\xi' \in \mathbb{R}^{n-1}$.

Obviously, t is smooth. One can also check that t is a trace symbol of order and class zero (using [18], (1.2.19) and (2.3.25), for example), defining therefore a trace operator $\tilde{T} : C_c^\infty(\bar{\mathbb{R}}_+^n) \to C_c^\infty(\mathbb{R}^{n-1})$. Moreover, t is polyhomogeneous and its homogeneous principal part is given by $t_0(x', \xi', \xi_n) = 2\pi\tilde{p}(x', \xi')\check{\varphi}(\xi_n/|\xi'|)$. We then get (using [18], (2.4.5) and (2.3.25), for example):

$$t_0(x', \xi', D_n) \;=\; |\xi'|\,\tilde{p}(x', \xi')\langle\varphi(|\xi'|\cdot)|. \tag{21}$$

Let D denote the linear span of all $p \otimes \langle\varphi|$, with $p \in C_c^\infty(S^*V)$, $\varphi \in C_c^\infty(\mathbb{R}_+)$. Since $C_c^\infty(\mathbb{R}_+)$ is dense in $L^2(\overline{\mathbb{R}}_+)$, D is dense in $C_0(S^*V, L^2(\overline{\mathbb{R}}_+)^*)$. The assignment

$$D \ni f(x', \xi') \mapsto \sqrt{|\xi'|}f(x', \xi') \circ \kappa_{|\xi'|} \tag{22}$$

($\kappa_{(\cdot)}$ as defined before Lemma 2), induces an isomorphism ι of $C_0(S^*V, L^2(\overline{\mathbb{R}}_+)^*)$ onto itself, since it is equal to the multiplication of an isometry by the function $\sqrt{|\xi'|}$, which is smooth, bounded, and bounded away from zero on S^*V.

Let $\rho \in C_c^\infty(\tilde{V})$, $\tilde{V} = \chi(V)$, be equal to one in a neighborhood of the x'-support of $\tilde{p}$. Then, extend χ to a chart χ_1 of X, and ρ to a cutoff ρ_1 with support in the image of χ_1. The pullback $T : C^\infty(X) \to C^\infty(\partial X)$ of $\rho\tilde{T}\rho_1$ by χ_1 is a trace operator and $\gamma(T) = \iota(p \otimes \langle\varphi|)$. Hence, $\operatorname{Im}\bar{\gamma}_{21}$ contains $\iota(D)$, which is dense in $C_0(S^*V, L^2(\overline{\mathbb{R}}_+)^*)$. On the other hand, $\operatorname{Im}\bar{\gamma}_{21}$ is closed, since $\bar{\gamma}$ is a C*-algebra homomorphism. $\Box$

The surjectivity of $\bar{\gamma}_{12}$ can be proven analogously to that of $\bar{\gamma}_{21}$. The main difference is that the homogeneous extension of p, which in the proof of Lemma 4 was of degree zero, now has to be taken of degree (-1). One should then define the local Poisson symbol by $k(x', \xi', \xi_n) = 2\pi\omega(|\xi'|)\tilde{p}(x', \xi')\check{\psi}(\xi_n/|\xi'|)$, with $\psi(t) = \varphi(-t)$; and replace $\langle\cdot|$ by $|\cdot\rangle$ in (21), and $f(x', \xi') \circ \kappa_{|\xi'|}$ by $\kappa_{|\xi'|} \circ f(x', \xi')$ on the right-hand side of (22).

We must now describe the upper-left corner of $\operatorname{Im}\bar{\gamma}$. As a first step, the following lemma will show that $\operatorname{Im}\bar{\gamma}_{11}$ contains $C(S^*\partial X) \otimes \mathfrak{T}_0$ ($\mathfrak{T}_0 = \ker\lambda$, as defined after

(7)). Before proving it, let us show that the C*-algebra (let us call it $\mathfrak{T}_1$, for the moment) generated by all $\varphi(D)_+$, $\varphi \in \mathcal{S}(\mathbb{R})$, is equal to $\mathfrak{T}_0$. Since every generator of $\mathfrak{T}_1$ clearly belongs to $\mathfrak{T}_0$, one gets at once that $\mathfrak{T}_1 \subseteq \mathfrak{T}_0$.

Given φ in the space $C_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ that vanish at infinity, $U^{-1}F\varphi(D)_+F^{-1}U$ is equal to the Toeplitz operator on $\mathbb{S}^1$ of symbol $\varphi(\frac{iz-i}{z+1})$ (recall that U was defined in (6) and F denotes the Fourier transform). The isomorphism $\mathfrak{T}/\mathfrak{K} \simeq C(\mathbb{S}^1)$ then implies that $\mathfrak{T}_0$ is equal to the set of all $\varphi(D)_+ + C$, with $\varphi \in C_0(\mathbb{R})$ and C compact. It follows from the estimate $||\varphi(D)_+|| = \sup|\varphi|$ that every $\varphi(D)_+$, with φ in $C_0(\mathbb{R})$, is the norm limit of a sequence $\varphi_k(D)_+$, $\varphi_k \in \mathcal{S}(\mathbb{R})$. This shows that $\mathfrak{T}_1$ is equal to the C*-algebra generated by all $\varphi(D)_+$, $\varphi \in C_0(\mathbb{R})$. To prove that $\mathfrak{T}_0 \subseteq \mathfrak{T}_1$, it is therefore enough to show that $\mathfrak{T}_1$ contains $\mathfrak{K}$. This follows from the fact ([11], Proposition 7.12) that $\mathfrak{K}$ is equal to the commutator ideal of $\mathfrak{T}$, which is equal to the commutator ideal of $\mathfrak{T}_1$. Indeed, the two ideals are equal to the closed linear span of all products $T_1 \cdots T_k$, where at least one (possibly more) of the T_j's is of the form $[\varphi(D)_+, \psi(D)_+]$, φ and ψ in $C_0(\mathbb{R})$, and the others are of the form $\varphi(D)_+$, φ in $C_0(\mathbb{R})$.

Lemma 5. *Let V be as in Lemma 4. Then $C_0(S^*V, \mathfrak{T}_0)$ is contained in $Im\,\bar{\gamma}_{11}$.*

Proof: Given $\varphi \in \mathcal{S}(\mathbb{R})$ and $p \in C_c^\infty(S^*V)$, let P denote the pullback by χ_1 of the pseudodifferential operator on $\bar{\mathbb{R}}^n_+$ of amplitude

$$q(x,y,\xi) = \omega(|\xi'|)\tilde{p}(x',\xi')\varphi(\xi_n/\sqrt{1+|\xi'|^2})\rho_1(x)\rho_1(y), \tag{23}$$

with χ, χ_1, ω, $\tilde{p}$ and ρ_1 as in the proof of Lemma 4. It follows from the proof of Lemma 5.3.1 in [40] that q is a classical amplitude, and that the corresponding homogeneous principal symbol $q_0(x,\xi)$ satisfies, for $x' \in \tilde{V}$ and $\xi \neq 0$, $q_0(x',0,\xi) = \tilde{p}(x',\xi')\varphi(\xi_n/|\xi'|)$. Moreover, P has the transmission property ([35], 2.2.2.1). We then get from (12):

$$\gamma(P_+)(x',\xi') \;=\; \tilde{p}(x',\xi') \cdot [\kappa_{|\xi'|} \circ \varphi(D)_+ \circ \kappa^{-1}_{|\xi'|}]. \tag{24}$$

It follows from the fact that $\mathfrak{T}_0$ is generated by all $\varphi(D)_+$, $\varphi \in \mathcal{S}(\mathbb{R})$, that $C_0(S^*V, \mathfrak{T}_0)$ is equal to the C*-algebra generated by all $\mathfrak{T}_0$-valued functions on S^*V as those at the right-hand side of (24), with $\varphi \in \mathcal{S}(\mathbb{R})$ and $p \in C_c^\infty(S^*V)$. $\square$

Now we show that $\bar{\gamma}$ is not surjective. We will regard $C(\partial X)$ as a subset of $C(S^*\partial X)$, and $C(S^*\partial X)$ as a subset of $C(S^*\partial X) \otimes \mathfrak{T}$ (recall that $\mathfrak{T}$ contains the identity operator I on $L^2(\overline{\mathbb{R}}_+)$).

Lemma 6. $\mathrm{Im}\,\bar{\gamma}_{11} \cap C(S^*\partial X) \;=\; C(\partial X)$.

Proof: Operators of multiplication by functions in $C^\infty(X)$ are the simplest examples of pseudodifferential operators with the transmission property on X. The boundary principal symbol of such an operator is equal to the restriction to the boundary of the multiplier (times I). This shows that $C^\infty(\partial X)$, and hence also $C(\partial X)$, are contained in $\mathrm{Im}\,\bar{\gamma}_{11}$.

Let f belong to $\mathrm{Im}\,\bar{\gamma}_{11} \cap C(S^*\partial X)$. For every $\delta > 0$, there exist a pseudodifferential operator with the transmission property P, and a singular Green operator G, such that $||\gamma(P_+)(x',\xi') + \gamma(G)(x',\xi') - f(x',\xi')|| < \delta$, for all $(x',\xi') \in S^*\partial X$ (we have used that the set of all $P_+ + G$ is dense in $\mathfrak{A}_{11}$). Since $\gamma(G)(x',\xi')$ is compact for every (x',ξ'), we get:

$$\inf_{C\in\mathfrak{K}} ||\gamma(P_+)(x',\xi') - f(x',\xi')I + C|| \;<\; \delta. \tag{25}$$

Using once more the isomorphism $\mathfrak{T}/\mathfrak{K} \simeq C(\mathbb{R} \cup \{\infty\})$, we get

$$\inf_{C \in \mathfrak{K}} ||p(D)_+ + C|| = \sup |p|,$$

for any $p \in \mathcal{H}_0$. Let p_0 denote the principal symbol of P, regarded as a zero-degree homogeneous function on the cotangent bundle, smooth except at the zero section. The left-hand side of (25) is then equal to

$$\sup_{\xi_n \in \mathbb{R}} |p_0(x', 0, \xi', \xi_n) - f(x', \xi')| = \sup_{\xi_n \neq 0} |p_0(x', 0, \frac{\xi'}{|\xi_n|}, \pm 1) - f(x', \xi')|,$$

which is greater than or equal to $|p_0(x', 0, 0, +1) - f(x', \xi')|$. This makes sense in view of our choice of a normal coordinate x_n after (2). Note also that, for covectors $\xi' \neq 0$, $p_0(x', 0, \xi', \cdot) \in \mathcal{H}_0$.

We have proven that, for all $\delta > 0$, there exists a $g \in C(\partial X)$ such that $\sup_{S^*\partial X} |f - g| < \delta$. Hence, $f \in C(\partial X)$. $\square$

The previous argument also shows that $p_0(x', 0, 0, +1) = p_0(x', 0, 0, -1)$. This is part of Boutet de Monvel's transmission condition for classical operators, see the comments before the statement of Theorem 5 for more details.

Lemmas 5 and 6 show that $\operatorname{Im} \bar{\gamma}_{11} \subseteq (C(S^*\partial X) \otimes \mathfrak{T}_0) \oplus C(\partial X)$. To prove the reverse inclusion, let us consider the C*-algebra homomorphism $1 \otimes \lambda$ from $C(S^*\partial X) \otimes \mathfrak{T}$ to itself that maps $f \otimes p(D)_+$ to $p(\infty) f \otimes I$, $p \in \mathcal{H}_0$. If F belongs to $\operatorname{Im} \bar{\gamma}_{11}$, then $F - (1 \otimes \lambda)(F)$ belongs to $C(S^*\partial X) \otimes \mathfrak{T}_0$. By Lemma 5, $F - (1 \otimes \lambda)(F)$, and therefore also $(1 \otimes \lambda)(F)$, belong to $\operatorname{Im} \bar{\gamma}_{11}$. By Lemma 6, $(1 \otimes \lambda)(F)$ is in $C(\partial X)$.

This proves the characterization of $\operatorname{Im} \bar{\gamma}$ promised after (7):

$$\operatorname{Im} \bar{\gamma} = \left(C(S^*\partial X) \otimes \begin{pmatrix} \mathfrak{T}_0 & L^2(\overline{\mathbb{R}}_+) \\ L^2(\overline{\mathbb{R}}_+)^* & \mathbb{C} \end{pmatrix} \right) \oplus \left(C(\partial X) \otimes \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix} \right). \tag{26}$$

Before stating the main result of this Section, however, a few more definitions and comments are needed. Let us denote by $\mathfrak{W}$ the C*-algebra of all the bounded operators $A = ((A_{jk}))_{j,k=1,2}$ on $L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$ such that $A_{11} \in \mathfrak{T}$, and by $\mathfrak{W}_0$ the set of all $A \in \mathfrak{W}$ such that $A_{11} \in \mathfrak{T}_0$ ($\mathfrak{W}$ is the closure of the set of all zero-order *Wiener-Hopf operators*, in Boutet de Monvel's terminology [4]). $\mathfrak{W}_0$ is clearly the kernel of the linear functional $\Lambda(A) = \lambda(A_{11})$; which is actually a homomorphism, since $\Lambda(AB) = \lambda(A_{11}B_{11} + A_{12}B_{21})$ and $A_{12}B_{21}$ is compact.

Temporarily denoting by $\mathfrak{K}_{\overline{\mathbb{R}}_+}$ the ideal of compact operators on $L^2(\overline{\mathbb{R}}_+)$ and by $\mathfrak{K}_\oplus$ that on $L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$, it is straightforward to check that the mapping that sends the class of A to the class of the matrix that has A in the upper-left corner and zero elsewhere is an isomorphism between $\mathfrak{T}/\mathfrak{K}_{\overline{\mathbb{R}}_+}$ and $\mathfrak{W}/\mathfrak{K}_\oplus$. The isomorphism $\mathfrak{T}/\mathfrak{K}_{\overline{\mathbb{R}}_+} \simeq C(\mathbb{S}^1)$ defined in the introduction therefore induces an isomorphism from $\mathfrak{W}/\mathfrak{K}_\oplus$ to $C(\mathbb{S}^1)$ that sends the class of the identity I to the constant function 1. The homomorphism Λ corresponds, then, to the evaluation at (-1) on $C(\mathbb{S}^1)$ composed with the projection of $\mathfrak{W}$ onto $\mathfrak{W}/\mathfrak{K}_\oplus$.

We have proven:

Theorem 2. *The C*-algebra $\mathfrak{W}$ contains the compact ideal, and $\mathfrak{W}/\mathfrak{K}$ and $C(\mathbb{S}^1)$ are isomorphic (as unital C*-algebras). Moreover, the image of $\bar{\gamma}$ is isomorphic to $C(\partial X) \oplus (C(S^*\partial X) \otimes \mathfrak{W}_0)$, with $\mathfrak{W}_0$ denoting the kernel of the homomorphism $\Lambda : \mathfrak{W} \to \mathbb{C}$ induced by the evaluation at (-1) on $C(\mathbb{S}^1)$.*

4. K-Theory

By a well-known consequence of Bott periodicity ([1], Theorem 9.3.1), to any short exact sequence of C*-algebras $0 \to J \to A \to A/J \to 0$, one may associate a cyclic six-term exact sequence of abelian groups

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array} , \tag{27}$$

where the horizontal arrows are functorially induced homomorphisms, the arrow connecting K_1 to K_0 is called the *index mapping*, and the other connecting mapping is called the *exponential mapping*. If J is the compact ideal, then $K_1(J) = 0$, $K_0(J) \simeq \mathbb{Z}$, and the index mapping is the Fredholm index ([1], 8.3.2).

Lemma 7. $K_i(C(S^*\partial X) \otimes \mathfrak{W}_0) = 0$, $i = 0, 1$.

Proof: The six-term exact sequence we get from the short exact sequence given by the isomorphism of our Theorem 2, $0 \to \mathfrak{K} \to \mathfrak{W} \to C(\mathbb{S}^1) \to 0$, is

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(\mathfrak{W}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_1(\mathfrak{W}) & \longleftarrow & 0. \end{array} \tag{28}$$

Because there exists a Toeplitz operator of Fredholm index one, there exists also an operator in $\mathfrak{W}$ of index one (take a matrix with 1 in the lower right corner and 0 outside the diagonal). Hence, the index mapping in (28) is surjective. This gives $K_0(\mathfrak{W}) = [I] \cdot \mathbb{Z}$ and $K_1(\mathfrak{W}) = 0$. It then follows from the six-term exact sequence associated to

$$0 \longrightarrow \mathfrak{W}_0 \longrightarrow \mathfrak{W} \xrightarrow{\Lambda} \mathbb{C} \longrightarrow 0$$

that $K_0(\mathfrak{W}_0) = K_1(\mathfrak{W}_0) = 0$. By Künneth's theorem for tensor products [1, 45], we prove our claim. $\square$

Denoting by the same letters functions on X or ∂X and the multiplication operators they define, let b denote the (unital) C*-algebra homomorphism

$$\begin{array}{rcl} b : C(\partial X) & \longrightarrow & \operatorname{Im} \bar{\gamma} \\ g & \longmapsto & \bar{\gamma}\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \right) \end{array} , \tag{29}$$

where f denotes a function in $C(X)$ whose restriction to the boundary equals g. The proof of Theorem 2 shows that $\operatorname{Im} \bar{\gamma} = (C(S^*\partial X) \otimes \mathfrak{W}_0) \oplus \operatorname{Im} b$. Since b is injective, the previous lemma implies:

Corollary 8. *The homomorphisms* $b_* : K_i(C(\partial X)) \to K_i(\operatorname{Im} \bar{\gamma})$, $i = 0, 1$, *induced by* b *are isomorphisms.*

Proposition 9. *If* X *is connected and* ∂X *is not empty, then there exists a non-vanishing section of the cotangent bundle of* X.

Proof: Let Σ be a section of the cotangent bundle of Ω with a finite number of zeros. Given $x_0 \in X$, a zero of Σ, let $c : [0,1] \to \Omega$ be a smooth curve with $c(0) = x_0$ and $c(1) \notin X$, such that $c(t)$ is not a zero of Σ if $t \neq 0$, and $c'(t) \neq 0$ for all t. Let V be a vector field on Ω such that $V(c(t)) = c'(t)$, $t \in [0,1]$, and $V \equiv 0$ outside a neighborhood of the image of c which intersects the set of zeros of Σ only at x_0. Let Φ_s, $s \in \mathbb{R}$, denote the flow of V, and write $\tilde{\Sigma}$ for the pushforward of Σ by Φ_1.

Comparing the zero sets of Σ and $\tilde{\Sigma}$, we see that one of the zeros of Σ has moved from X to its complement in Ω. After repeating this procedure a finite number of times, we are finished. $\square$

Corollary 10. *If, in addition, X is orientable and has dimension two, then the cotangent bundle of the interior, $T^*\dot{X}$, is homeomorphic to $\dot{X} \times \mathbb{R}^2$.*

Proof: Starting with the section given by Proposition 9, one may use orientability to get a smooth frame for the cotangent bundle. $\square$

Let $m' : C_0(\dot{X}) \to C_0(S^*\dot{X})$ denote the pullback of functions under the bundle projection $S^*\dot{X} \to \dot{X}$. Proposition 9 makes it possible to choose a continuous section Σ of the cosphere bundle. Let then $s : C_0(S^*\dot{X}) \to C_0(\dot{X})$ denote the C*-algebra homomorphism $f \mapsto f \circ \Sigma$. It is clear that $s \circ m'$ is the identity on $C_0(\dot{X})$.

Let $B^*\dot{X}$ denote the bundle of closed unit balls of the cotangent bundle of the interior. The kernel of the restriction mapping $R : C_0(B^*\dot{X}) \to C_0(S^*\dot{X})$ is homeomorphic to $C_0(T^*\dot{X})$. This observation defines the exact sequence

$$0 \to C_0(T^*\dot{X}) \to C_0(B^*\dot{X}) \to C_0(S^*\dot{X}) \to 0. \tag{30}$$

Proposition 11. *If X is connected and ∂X is not empty, then, for each $i = 0, 1$, $K_i(C_0(S^*\dot{X}))$ is isomorphic to $K_i(C_0(\dot{X})) \oplus K_{1-i}(C_0(T^*\dot{X}))$. Under this isomorphism, the homomorphism induced by m' corresponds to the canonical injection of $K_i(C_0(\dot{X}))$ into $K_i(C_0(\dot{X})) \oplus K_{1-i}(C_0(T^*\dot{X}))$, and the connecting mapping $K_i(C_0(S^*\dot{X})) \to K_{1-i}(C_0(T^*\dot{X}))$ in the six-term exact sequence one gets from (30) corresponds to the canonical projection of $K_i(C_0(\dot{X})) \oplus K_{1-i}(C_0(T^*\dot{X}))$ onto $K_{1-i}(C_0(T^*\dot{X}))$.*

Proof: Since closed balls can be continuously deformed to their centers, the pullback of functions under the bundle projection $m'' : C_0(\dot{X}) \to C_0(B^*\dot{X})$ is a homotopy equivalence. Hence, the induced homomorphisms, $m''_* : K_i(C_0(\dot{X})) \to K_i(C_0(B^*\dot{X}))$, $i = 0, 1$, are isomorphisms ([45], 6.4.3 and 7.1.6). It is obvious that $R \circ m'' = m'$. Hence, if we use the isomorphisms m''_* to identify $K_i(C_0(B^*\dot{X}))$ and $K_i(C_0(\dot{X}))$, the six-term exact sequence associated to (30) becomes

$$\begin{array}{ccccc} K_0(C_0(T^*\dot{X})) & \longrightarrow & K_0(C_0(\dot{X})) & \xrightarrow{m'_*} & K_0(C_0(S^*\dot{X})) \\ \uparrow & & & & \downarrow \\ K_1(C_0(S^*\dot{X})) & \xleftarrow{m'_*} & K_1(C_0(\dot{X})) & \longleftarrow & K_1(C_0(T^*\dot{X})) \end{array}. \tag{31}$$

We have seen, right after Corollary 10, that $s \circ m'$ is the identity. That implies that s_* is a left inverse for m'_*. The cyclic sequence (31) then becomes two split exact sequences

$$0 \longrightarrow K_i(C_0(\dot{X})) \underset{s_*}{\overset{m'_*}{\leftrightarrows}} K_i(C_0(S^*\dot{X})) \longrightarrow K_{1-i}(C_0(T^*\dot{X})) \longrightarrow 0,$$

$i = 0, 1$. That the above sequences also split on the right and the connecting mappings correspond to projections follows now from algebraic generalities ([45], 3.1.4, for example). $\square$

Theorem 3. *If X is connected and ∂X is not empty, then, for each $i = 0, 1$, there is an exact sequence*

$$0 \longrightarrow \ker r_*^i \oplus K_{1-i}(C_0(T^*\dot{X})) \longrightarrow K_i(\mathfrak{A}/\mathfrak{K}) \longrightarrow \operatorname{Im} r_*^i \longrightarrow 0, \tag{32}$$

where $r_*^i : K_i(C(X)) \to K_i(C(\partial X))$ is the homomorphism induced by the restriction to the boundary $r : C(X) \to C(\partial X)$.

Proof: Let us consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathfrak{J}/\mathfrak{K} & \longrightarrow & \mathfrak{A}/\mathfrak{K} & \xrightarrow{\pi} & \mathfrak{A}/\mathfrak{J} & \longrightarrow 0 \\ & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle b} & \\ 0 \longrightarrow & C_0(\dot{X}) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) & \longrightarrow 0 \end{array}, \tag{33}$$

where m is the isometric *-homomorphism that maps $f \in C(X)$ to the class of $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$, g denoting the restriction to ∂X of f; recall that b was defined in (29) (here we do not distinguish between the isomorphic C*-algebras $\mathfrak{A}/\mathfrak{J}$ and $\operatorname{Im}\bar{\gamma}$).

Let us denote by δ and exp the index and exponential mappings associated to the top exact sequence in (33), and by by δ^0 and $\exp^0$ the index and exponential mappings associated to the bottom one. By the naturality of the connecting mappings [37], we get from (33) the two commutative diagrams

$$\begin{array}{ccc} K_1(\mathfrak{A}/\mathfrak{J}) & \xrightarrow{\delta} & K_0(\mathfrak{J}/\mathfrak{K}) \\ \uparrow{\scriptstyle b_*} & & \uparrow{\scriptstyle m_*} \\ K_1(C(\partial X)) & \xrightarrow{\delta^0} & K_0(C_0(\dot{X})) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_0(\mathfrak{A}/\mathfrak{J}) & \xrightarrow{\exp} & K_1(\mathfrak{J}/\mathfrak{K}) \\ \uparrow{\scriptstyle b_*} & & \uparrow{\scriptstyle m_*} \\ K_0(C(\partial X)) & \xrightarrow{\exp^0} & K_1(C_0(\dot{X})) \end{array}. \tag{34}$$

The C*-algebra homomorphism m', defined after Corollary 10, is the composition of the isomorphism $j : \mathfrak{J}/\mathfrak{K} \to C_0(S^*\dot{X})$ of Theorem 1 with m, defined in (33). Inserting the isomorphisms $j_* : K_i(\mathfrak{J}/\mathfrak{K}) \to K_i(C_0(S^*\dot{X}))$ into the upper right corners of the diagrams in (34), and denoting by δ' and $\exp'$ the compositions of j_* with δ and exp, we get:

(35)

$$\begin{array}{ccc} K_1(\mathfrak{A}/\mathfrak{J}) & \xrightarrow{\delta'} & K_0(C_0(S^*\dot{X})) \\ \uparrow{\scriptstyle b_*} & & \uparrow{\scriptstyle m'_*} \\ K_1(C(\partial X)) & \xrightarrow{\delta^0} & K_0(C_0(\dot{X})) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_0(\mathfrak{A}/\mathfrak{J}) & \xrightarrow{\exp'} & K_1(C_0(S^*\dot{X})) \\ \uparrow{\scriptstyle b_*} & & \uparrow{\scriptstyle m'_*} \\ K_0(C(\partial X)) & \xrightarrow{\exp^0} & K_1(C_0(\dot{X})) \end{array}.$$

In other words, modulo isomorphisms and split injections, the connecting mappings in the six-terms exact sequences associated to the two horizontal short exact sequences in (33) are the same. That reduces the computation of the connecting mappings in the six-term exact sequence associated (with the use of the isomorphisms b_* and j_*) to $0 \to \mathfrak{J}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{J} \to 0$,

$$\begin{array}{ccccc} K_0(C_0(S^*\dot{X})) & \longrightarrow & K_0(\mathfrak{A}/\mathfrak{K}) & \longrightarrow & K_0(C(\partial X)) \\ \uparrow & & & & \downarrow \\ K_1(C(\partial X)) & \longleftarrow & K_1(\mathfrak{A}/\mathfrak{K}) & \longleftarrow & K_1(C_0(S^*\dot{X})) \end{array}, \tag{36}$$

to a purely topological operation.

Using Proposition 11 and (35) to compare the cyclic exact sequences (36) and the one we get from r,

$$\begin{array}{ccccc} K_0(C_0(\dot{X})) & \longrightarrow & K_0(C(X)) & \longrightarrow & K_0(C(\partial X)) \\ \uparrow{\scriptstyle \delta^0} & & & & \downarrow{\scriptstyle \exp^0} \\ K_1(C(\partial X)) & \longleftarrow & K_1(C(X)) & \longleftarrow & K_1(C_0(\dot{X})) \end{array} , \tag{37}$$

we then obtain the two exact sequences in (32) by taking quotients by the kernels and restricting the ranges of the horizontal arrows in (36). □

It is well known that the K-groups of $C_0(Y)$ are finitely generated, for any manifold Y. That follows by induction, using triangularizability, starting from the fact that $K_0(C_0(Y)) \simeq \mathbb{C}$ and $K_1(C_0(Y)) = 0$ when Y is a point.

Corollary 12. *If we suppose, in addition to the hypothesis of Theorem 3, that the K-groups of $C(X)$, $C(\partial X)$ and $C_0(T^*\dot{X})$ are all torsion free, then $K_i(\mathfrak{A}/\mathfrak{K}) \simeq K_i(C(X)) \oplus K_{1-i}(C_0(T^*\dot{X}))$, $i = 0, 1$.*

Proof: By our previous remark, all groups appearing in the extremes of (32) are finitely generated. Our torsion-free hypothesis then implies that they are powers of $\mathbb{Z}$. For each $i = 0, 1$, the quotient of $K_i(\mathfrak{A}/\mathfrak{K})$ by a finitely generated free subgroup is therefore free and finitely generated. Hence, $K_i(\mathfrak{A}/\mathfrak{K})$ is also a power of $\mathbb{Z}$, isomorphic to $\ker r_*^i \oplus K_{1-i}(C_0(T^*\dot{X})) \oplus \operatorname{Im} r_*^i$. The same argument shows that $K_i(C(X))$ is also a power of $\mathbb{Z}$, isomorphic to $\ker r_*^i \oplus \operatorname{Im} r_*^i$. □

In Section 6, we apply Corollary 12 to orientable surfaces. We end this section showing how one can get the K-theory of $\mathfrak{A}$ and $\mathcal{A}$ from the K-theory of $\mathfrak{A}/\mathfrak{K}$. The result about K_1 in Proposition 13 is improved in Corollary 20.

Proposition 13. *The projection $\mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ induces isomorphisms from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{A}/\mathfrak{K})$, and from $K_1(\mathfrak{A})$ to the kernel of the Fredholm-index mapping $K_1(\mathfrak{A}/\mathfrak{K}) \to \mathbb{Z}$, which is surjective.*

Proof: As in the proof of Lemma 7, it suffices to prove that there exist an integer k and a k-by-k matrix with entries in $\mathfrak{A}$ which, regarded as an operator on $\mathfrak{H}^k$, is Fredholm and has index one.

It follows from Fedosov's index formula, as in the proof of Theorem 5.18 in [28], that there exists a k-by-k matrix S of zero-order pseudodifferential operators on ∂X defining an index-one Fredholm operator on $(H^{-\frac{1}{2}}(\partial X))^k$. The Green operator $\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$, I denoting the identity on $C^\infty(X; \mathbb{C}^k)$, is then a Fredholm operator of index one on $\mathfrak{H}^k$. □

Proposition 14. *The injection of $\mathcal{A}$ into $\mathfrak{A}$ induces isomorphisms between $K_i(\mathcal{A})$ and $K_i(\mathfrak{A})$, $i = 0, 1$.*

Proof: By [38], Corollary 4.11, $\mathcal{A}$ can be given the structure of a Fréchet *-algebra, such that the embedding of $\mathcal{A}$ in $\mathfrak{A}$ is continuous. Moreover, $\mathcal{A}$ contains the inverses of all its elements which are invertible in $\mathfrak{A}$. In particular, the set of invertibles in $\mathcal{A}$ is open, with respect to that Fréchet topology. Then, by [44], pg 115, the inversion is continuous. Hence, the Cauchy integrals that give the holomorphic functional calculus converge also in $\mathcal{A}$. Being closed under the holomorphic functional calculus, $\mathcal{A}$ has the same K-theory as $\mathfrak{A}$ ([2], Théorème A.2.1). □

5. A better result for K_1

Throughout this section, we assume that X is connected and that ∂X is not empty, in order to be able to apply Proposition 11 and the proof of Theorem 3.

As a first step, let us show how the principal symbol is related to the index mapping in the six-term exact sequence associated to (30). We need topological K-theory and refer to the first section of [1] for definitions and notation.

Let Y be a locally compact Hausdorff space, Z a closed subspace of Y, and $U = Y \setminus Z$. Let $\imath : K_0(C_0(U)) \to K(Y, Z)$ denote the composition of the isomorphism from $K(U)$ to $K(Y, Z)$ given in [1], 1.5.1, with the canonical isomorphism from $K_0(C_0(U))$ to $K(U)$.

We learned the following proof from Ruy Exel.

Proposition 15. *Let $\delta : K_1(C_0(Z)) \to K_0(C_0(U))$ be the index mapping in the six-term exact sequence associated to $0 \to C_0(U) \to C_0(Y) \to C_0(Z) \to 0$. Then $\imath \circ \delta : K_1(C_0(Z)) \to K(Y, Z)$ maps the class of an invertible k-by-k matrix $u \in M_k(C_0(Z)^+)$, $C_0(Z)^+ = C(Z^+) = C(Z \cup \{+\})$, to the class of the triple (E^k, E^k, u), where E^k denotes the rank-k trivial bundle and u is regarded as an isomorphism on the restriction of E^k to Z in the obvious way.*

Proof: We may suppose, without loss of generality, that $u(+)$ is the identity. Let $w \in M_{2k}(C_0(Y)^+)$ be an invertible coinciding with $u \oplus u^{-1} \in M_{2k}(C_0(Z)^+)$ on Z and such that $w(+)$ is the identity. By definition ([45], 8.1.1), $\delta([u]) = [wp_k w^{-1}] - [p_k]$, where p_k is the $2k$-by-$2k$ matrix with 1 on the first k entries of the diagonal and zero elsewhere. The above mentioned canonical isomorphism maps an element $[p] - [q] \in K_0(C_0(U))$, p and q idempotents in $M_l(C_0(U)^+)$, to the element of $K(U) \simeq K(U^+, +)$ defined by the triple $(\operatorname{Im} p, \operatorname{Im} q, \alpha)$, with $\operatorname{Im} p$ denoting the vector bundle $\{(x, v) \in U^+ \times \mathbb{C}^l ; v \in \operatorname{Im} p(x)\}$, and α being any isomorphism between $\operatorname{Im} p(+)$ and $\operatorname{Im} q(+)$ (*any* homomorphism because the equivalence class of this triple depends only on the homotopy class of α, by [24], II.2.15, and the complex linear group $\mathrm{Gl}_k(\mathbb{C})$ is connected; this also justifies the first statement in this proof). Since $wp_k w^{-1}$ is equal to p_k at the infinite point $+$, the fibers of $\operatorname{Im} wp_k w^{-1}$ and $\operatorname{Im} p_k$ are canonically isomorphic there. Hence, viewed as an element of $K(U)$, $\delta([u])$ is the class of the triple $(\operatorname{Im} wp_k w^{-1}, \operatorname{Im} p_k, \mathrm{id})$, where id denotes the identity mapping.

Let $\phi : Y^+ \to U^+$ be the identity on U and map all other points to $+$. The isomorphism from $K(U)$ to $K(Y, Z)$ defined in [1], 1.5.1, is the homomorphism contravariantly induced by ϕ, viewed as a morphism between the compact pairs (Y^+, Z^+) and $(U^+, +)$. Hence, it maps $\delta([u])$ to the class of the triple $(\operatorname{Im} wp_k w^{-1}, \operatorname{Im} p_k, \mathrm{id})$ (all that ϕ^* does is to regard the projections defining the triple as functions on Y^+ which are constant over Z^+).

The mapping $f : \operatorname{Im} p_k \to \operatorname{Im} wp_k w^{-1}$, $f(x, v) = (x, w(x)v)$, is a vector bundle isomorphism. Moreover, f and the identity mapping on $\operatorname{Im} p_k$ intertwine $u \oplus u^{-1}$ and the identity. The triple $(\operatorname{Im} wp_k w^{-1}, \operatorname{Im} p_k, \mathrm{id})$ is therefore equivalent to $(\operatorname{Im} p_k, \operatorname{Im} p_k, u \oplus u^{-1})$, which is obviously equivalent to (E^k, E^k, u). □

We have seen in Proposition 11 that $K_1(C_0(S^*\dot{X}))$ is isomorphic to $K_1(C_0(\dot{X})) \oplus K_0(C_0(T^*\dot{X}))$, with the canonical projection onto $K_0(C_0(T^*\dot{X}))$ corresponding to the index mapping $\delta^1 : K_1(C_0(S^*\dot{X})) \to K_0(C_0(T^*\dot{X}))$ associated to (30). Applying Proposition 15, with $Y = B^*\dot{X}$ and $Z = S^*\dot{X}$, we may give another description of δ^1, now regarded as a mapping from $K_1(\mathfrak{I}/\mathfrak{K})$ to $K_0(C_0(T^*\dot{X}))$ (by

Theorem 1). Let $[[A]] \in K_1(\mathfrak{I}/\mathfrak{K})$ be given, $[[A]]$ denoting the K_1-class of the class $[A] \in (\mathfrak{I}/\mathfrak{K})^+ \subset \mathfrak{A}/\mathfrak{K}$ of a Fredholm operator $A \in M_k(\mathfrak{I} \oplus \mathbb{C})$ with principal symbol $\bar{\sigma}(A)$ (we denote by $\mathfrak{C}^+$ the unitization of a C*-algebra $\mathfrak{C}$). $\delta^1([[A]])$ is then equal to the class of the triple $(E^k, E^k, \bar{\sigma}(A))$ in $K(B^*\dot{X}, S^*\dot{X}) \simeq K(T^*\dot{X}) \simeq K_0(C_0(T^*\dot{X}))$.

In our language, part of the content of Boutet de Monvel's index theorem is that $\delta^1 : K_1(C_0(S^*\dot{X})) \to K_0(C_0(T^*\dot{X}))$ factors through $K_1(\mathfrak{A}/\mathfrak{K})$. More precisely, let us denote by $\mathtt{ind} : K_1(\mathfrak{A}/\mathfrak{K}) \to K_0(C_0(T^*\dot{X}))$ the composition of the canonical isomorphism between $K(T^*\dot{X})$ and $K_0(C_0(T^*\dot{X}))$ with the homomorphism defined in [4], Theorem 5.21 (see also [35], 3.2.2.4, Theorem 1). We then have:

Lemma 16. *With i_* denoting the homomorphism induced by the inclusion of $\mathfrak{I}/\mathfrak{K}$ into $\mathfrak{A}/\mathfrak{K}$, one has $\mathtt{ind} \circ i_* = \delta^1$.*

Proof: Let $x = [[A]] \in K_1(\mathfrak{I}/\mathfrak{K})$ be given, A as above. Since the scalar part of A is invertible and $\mathrm{Gl}_k(\mathbb{C})$ is connected, we may suppose that $A = B + I$, for some $B \in M_k(\mathfrak{I})$ and I the k-by-k identity matrix. By definition of $\mathfrak{I}$, there is a sequence $B_j \to B$, $B_j = \begin{pmatrix} \varphi P\psi + G & K \\ T & S \end{pmatrix} \in \mathcal{I}$, with P, G, etc, denoting k-by-k matrices of operators as those in (5). For sufficiently large j, then, $x = [[I + B_j]]$ (by [37], 2.1.11). Since $\begin{pmatrix} G & K \\ T & S \end{pmatrix}$ is compact, $x = \left[\left[\begin{pmatrix} \varphi P\psi + I & 0 \\ 0 & I \end{pmatrix}\right]\right]$, with I denoting the identity operators on $C^\infty(X)$ and on $C^\infty(\partial X)$. This operator is in the form [4], (5.21)-(3). According to Boutet de Monvel's prescription, $\mathtt{ind}\,(x)$ corresponds to the class in $K(T^*\dot{X}) \simeq K(B^*\dot{X}^+, S^*\dot{X}^+)$ determined by the principal symbol of $\varphi P\psi + I$ (the contribution from the identity on $C^\infty(\partial X)$ vanishes). This proves the lemma, by our comments after the proof of Proposition 15. □

Using Lemma 16, Proposition 11, and the proof of Theorem 3, we then obtain the commutative diagram:

$$
\begin{array}{cccccccc}
 & K_0(C_0(T^*\dot{X})) & & & & & & \\
 & \uparrow \delta^1 & & \nwarrow \mathtt{ind} & & & & \\
(38) & K_1(C_0(\dot{X})) \oplus K_0(C_0(T^*\dot{X})) & \xrightarrow{i_*} & K_1(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\pi_*} & K_1(\mathfrak{A}/\mathfrak{I}) & \xrightarrow{\delta'} & . \\
 & \uparrow m'_* & & \uparrow m_* & & \uparrow b_* & & \\
 & K_1(C_0(\dot{X})) & \xrightarrow{i'_*} & K_1(C(X)) & \xrightarrow{r_*} & K_1(C(\partial X)) & \xrightarrow{\delta^0} &
\end{array}
$$

Lemma 17. *In the diagram, above, we have $\mathtt{ind} \circ m_* = 0$.*

Proof: We want to show that $\mathtt{ind}\,(x) = 0$, whenever $x \in K_1(\mathfrak{A}/\mathfrak{K})$ is of the form $x = [[A]]$, with $A = \begin{pmatrix} f & 0 \\ 0 & f|_{\partial X} \end{pmatrix}$, for some invertible $f \in M_k(C(X))$. Such an A is in the form of [35], 3.2.2.4, Theorem 1, (iii). Indeed, the symbol f of its upper left corner is independent of the covariable not only on a neighborhood of the boundary, but over all X. The class of the triple (E^k, E^k, f) in $K(B^*X, S^*X \cup T^*X|_{\partial X}) \simeq K(T^*\dot{X})$ is zero, because f defines a bundle isomorphism of $E^k = B^*X \times \mathbb{C}^k$ onto itself. For the same reason, the element of $K(T^*\partial X)$ determined by $f|_{\partial X}$ also vanishes. Hence, $\mathtt{ind}\,(x) = 0$. □

Theorem 4. *If X is connected and ∂X is not empty, then $K_1(\mathfrak{A}/\mathfrak{K})$ is isomorphic to $K_1(C(X)) \oplus K_0(C_0(T^*\dot{X}))$. More precisely, in the diagram (38), m_* and the* *restriction of i_* to $K_0(C_0(T^*\dot{X}))$ are injective, and $K_1(\mathfrak{A}/\mathfrak{K}) = m_*(K_1(C(X)) \oplus i_*(K_0(C_0(T^*\dot{X})))$.*

Proof: Given $x \in K_1(\mathfrak{A}/\mathfrak{K})$, we have $\delta^0(b_*^{-1}(\pi_*(x))) = 0$, because $\delta'(\pi_*(x)) = 0$, m'_* is injective on $K_0(C_0(\dot{X}))$, and the left diagram in (35) commutes. It is therefore possible to choose $y \in K_1(C(X))$ such that $r_*(y) = b_*^{-1}(\pi_*(x))$. Since $\pi_*(x - m_*(y)) = 0$, there exists $z_1 \oplus z_2 \in K_1(C_0(\dot{X})) \oplus K_0(C_0(T^*\dot{X}))$ such that $i_*(z_1 \oplus z_2) = x - m_*(y)$.

In Proposition 11, we showed that m'_* is the canonical injection of $K_1(C_0(\dot{X}))$ into $K_1(C_0(\dot{X})) \oplus K_0(C_0(T^*\dot{X}))$. The commutativity of the lower left subdiagram in (38) then implies that $x = m_*(i'_*(z_1)) + m_*(y) + i_*(z_2)$. This proves that $m_*(K_1(C(X)) + i_*(K_0(C_0(T^*\dot{X}))) = K_1(\mathfrak{A}/\mathfrak{K})$.

To show that the intersection of $m_*(K_1(C(X))$ and $i_*(K_0(C_0(T^*\dot{X})))$ is 0, let us suppose that $m_*(y) = i_*(0 \oplus z)$, for some $y \in K_1(C(X))$ and $z \in K_0(C_0(T^*\dot{X}))$. Since $\mathtt{ind}\,(m_*(y)) = 0$ (Lemma 17), $z = \mathtt{ind}\,(i_*(0 \oplus z)) = \mathtt{ind}\,(m_*(y)) = 0$.

The existence of the homomorphism `ind` implies at once that i_* restricted to $K_0(C_0(T^*\dot{X}))$ is injective: $\mathtt{ind}\,(i_*(0 \oplus z)) = z$.

To prove that m_* is injective, let an invertible $f \in M_k(C(X))$ be given, such that $A = \begin{pmatrix} f & 0 \\ 0 & f|_{\partial X} \end{pmatrix}$ can be connected to the identity by a continuous path of Fredholm operators in $M_k(\mathfrak{A})$. Using that $[A] \mapsto \bar{\sigma}(A)$ defines a continuous mapping from $M_k(\mathfrak{A}/\mathfrak{K})$ to $M_k(C(S^*X))$ (this follows from (4)), we then get a homotopy of invertibles in $M_k(C(S^*X))$, between the k-by-k identity matrix I and f (regarded as a function on S^*X independent of the covarible). In Proposition 9, we showed that there exists a continuous section Σ of S^*X. By composition with Σ, any homotopy in $M_k(C(S^*X))$ defines a homotopy in $M_k(C(X))$. Hence, there is a homotopy of invertibles in $M_k(C(X))$ connecting f and I. The class defined by f in $K_1(C(X))$ therefore vanishes. □

The following two corollaries follow immediately from Lemma 17 and Theorem 4.

Corollary 18. *With respect to the isomorphism of Theorem 4,* `ind` *corresponds to the canonical projection from $K_1(C(X)) \oplus K_0(C_0(T^*\dot{X}))$ onto $K_0(C_0(T^*\dot{X}))$.*

Corollary 19. $0 \longrightarrow K_1(C(X)) \xrightarrow{m_*} K_1(\mathfrak{A}/\mathfrak{K}) \xrightarrow{\mathtt{ind}} K_0(C_0(T^*\dot{X})) \longrightarrow 0$ *is exact.*

This section was inspired by conversations with Anton Savin about Boutet de Monvel's index theorem, at conferences in Potsdam and Będlewo. Corollary 19 is his conjecture.

Next we show that also $K_1(\mathfrak{A})$ is topologically determined. We are going to use that there exists a mapping $\chi : K_0(C_0(T^*\dot{X})) \to \mathbb{Z}$ (the *topological index*) such that $\chi \circ \mathtt{ind}$ gives the Fredholm index ([4], Section 5.8; [35], 3.2.2.3, 3.2.2.4).

Corollary 20. $K_1(\mathfrak{A})$ *and* $K_1(\mathcal{A})$ *are isomorphic to* $K_1(C(X)) \oplus \ker\chi$.

Proof: It follows from Boutet de Monvel's index theorem, quoted above, and from our Proposition 13, that $K_1(\mathfrak{A})$ is isomorphic to $\ker(\chi \circ \mathtt{ind})$. Theorem 4 and Corollary 18 imply that an arbitrary element of $K_1(\mathfrak{A}/\mathfrak{K})$ is of the form $m_*(x) \oplus i_*(y)$, $x \in K_1(C(X))$ and $y \in K_0(C_0(T^*\dot{X}))$, and that $\mathtt{ind}\,(m_*(x) \oplus i_*(y)) = y$. Then it is obvious that $\chi \circ \mathtt{ind}\,(m_*(x) \oplus i_*(y)) = 0$ if and only if $y \in \ker\chi$. In Theorem 4, we also proved that m_* is injective. □

6. Orientable Surfaces

Throughout this section, X denotes a connected orientable two-dimensional manifold with nonempty boundary ∂X. The genus of X is denoted by g and the number of connected components of ∂X by m.

It is probably well known (we thank Thomas Schick for this information) that any manifold like our X can be continuously deformed to the union of q circles with one point in common, $q = 2g + m - 1$. That already implies that $K_0(C(X)) \simeq \mathbb{Z}$ and $K_1(C(X)) \simeq \mathbb{Z}^q$. In the proof of Proposition 21, we give a precise description of that deformation, which is also used to prove Proposition 22.

Proposition 21. *There exist $V \in X$, closed curves $C_1, \cdots, C_q$ in X, $q = 2g+m-1$, such that $C_j \cap C_k = \{V\}$ if $j \neq k$, and an isomorphism from $\mathbb{Z}^q$ to $K_1(C(X))$, which maps each element e_k of the canonical basis to the class of a unitary in $C(X)$ equal to one on C_j, if $j \neq k$, and with winding number one on C_k. Moreover, $K_0(C(X)) = [1] \cdot \mathbb{Z}$, where $[1]$ denotes the class of the function identical to 1 on X.*

Proof: Any closed orientable surface of genus g is homeomorphic to a polygon of $4g$ sides, identified in pairs, all vertices corresponding to the same point in the manifold ([15], Section 17b). Orientability implies that, if l and l' are two identified sides, then the polygon is to the left of l if and only if it is to the right of l' (assuming, of course, that the parametrizations of l and l' are the same, with respect to the identification). Our surface X is then homeomorphic to such a polygon with sides identified, with m disjoint open disks, $D_1, \cdots, D_m$, removed from its interior. Let us choose a side $l = [P, Q]$ in this polygon and draw $m-1$ curves, $c_1, \cdots, c_{m-1}$, all going from P to Q, so that D_1 is between l and c_1; D_k is between c_{k-1} and c_k, $k = 2, \cdots, m-1$; and D_m is between c_{m-1} and the remaining sides.

Each curve c_k and each pair of identified sides in the polygon correspond to circles (closed curves) in X that do not intersect ∂X. Let us denote by C_k the circles obtained from c_k, $k = 1, \cdots, m-1$; by C_m the circle that comes from l and its pair; and by $C_{m+1}, \cdots, C_q$ the circles that come from the other sides of the polygon. Any two among these q circles meet in exactly one point, V, the equivalence class of the vertices of the polygon. Let us denote by Y the union of the circles $C_1, \cdots, C_q$. Gradually enlarging the disks $D_1, \cdots, D_m$, without crossing any of the C_k's, but with their boundaries eventually adhering to them, one proves that X is homotopically equivalent to Y.

Looking at what this deformation does to continuous functions on X, one proves that the restriction mapping $R : C(X) \to C(Y)$ is a homotopy equivalence. Hence, we get the isomorphisms

$$R_* : K_i(C(X)) \to K_i(C(Y)),\ i = 0, 1. \tag{39}$$

For any circle C containing a point V, let us denote by $C_0(C\backslash V)$ the algebra of continuous functions on C that vanish at V. We are going to use that $K_0(C_0(C\backslash V)) = 0$; and that $K_1(C_0(C\backslash V)) \simeq \mathbb{Z}$, with isomorphism given by the winding number. Moreover, there exists a generator of $K_1(C_0(C\backslash V))$ which is equal to one at V.

Let us consider the exact sequence

$$0 \longrightarrow \bigoplus_{k=1}^{q} C_0(C_k\backslash V) \xrightarrow{i} C(Y) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0, \tag{40}$$

where i denotes the inclusion mapping, and π evaluation at V. Using that $K_0(\mathbb{C}) \simeq \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, we get from (40):

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(Y)) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(Y)) & \longleftarrow & \mathbb{Z}^q \end{array} . \tag{41}$$

The exponential mapping in (41) is the zero map, since the upper-right horizontal arrow in maps [1] to $1 \in \mathbb{Z}$. We then get that $K_0(C(Y))$ is isomorphic to $\mathbb{Z}$, and the statement about K_0 follows from (39). We also get that the inclusion mapping in (40) induces an isomorphism i_* from $\oplus_k K_1(C_0(C_k \backslash V)) \simeq \mathbb{Z}^q$ to $K_1(C(Y))$. The composition $R_*^{-1} \circ i_*$ gives the other isomorphism for which we were looking. □

$K_0(C(\partial X))$ and $K_1(C(\partial X))$ are both isomorphic to $\mathbb{Z}^m$. The generators of $K_0(C(\partial X))$ are the classes of the functions equal to one on the k-th connected component of ∂X (the boundary of D_k, as defined above), and zero on the others. We choose the isomorphism from $K_1(C(\partial X))$ to $\mathbb{Z}^m$ given by the winding number, with respect to the orientation of ∂X induced by the orientation of X.

Let us now consider the exact sequence $0 \to C_0(\dot{X}) \to C(X) \to C(\partial X) \to 0$ induced by the restriction to the boundary. Detailed information about all the homomorphisms in the corresponding six-term exact sequence,

$$\begin{array}{ccccc} K_0(C_0(\dot{X})) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^m \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^m & \longleftarrow & \mathbb{Z}^q & \longleftarrow & K_1(C_0(\dot{X})) \end{array} , \tag{42}$$

is given in the following proposition.

Proposition 22. *$K_0(C_0(\dot{X}))$ is isomorphic to $\mathbb{Z}$ and $K_1(C_0(\dot{X}))$ is isomorphic to $\mathbb{Z}^q$. The index mapping in (42) is surjective and, with respect to the previously defined isomorphisms, has kernel equal to $\{(j_1, \cdots, j_m); \sum j_k = 0\}$. Moreover, the exponential mapping has kernel generated by $(1, \cdots, 1)$ and the lower-right arrow in (42) has image isomorphic to $\mathbb{Z}^{2g}$, generated by $e_{m+1}, \cdots, e_q$ and $(e_1 + \cdots + e_m)$.*

Proof: The upper-right horizontal arrow in (42) maps $1 \in \mathbb{Z}$ to the nonzero element $([1], \cdots, [1])$ of $K_0(C(\partial X))$. That means that the upper-left arrow is the zero mapping and that the kernel of the exponential is $(1, \cdots, 1) \cdot \mathbb{Z}$.

Paying close attention to how the connected components of ∂X touch the C_k's, at the end of the deformation described in the proof of Proposition 21, we see that the lower-left horizontal arrow in (42) maps e_k to $e_{k+1} - e_k$, $k = 1, \cdots, m-1$; e_m to $e_1 - e_m$; and the remaining $e_{m+1}, \cdots e_q$ to zero. We then get that the kernel of the index mapping is what we stated, and that the kernel of the lower-left arrow in (42) is generated by $e_{m+1}, \cdots, e_q$ and $(e_1 + \cdots + e_m)$. Moreover, the quotient of $\mathbb{Z}^m$ by the kernel of the index mapping is isomorphic to $\mathbb{Z}$, what proves our statement about $K_0(C_0(\dot{X}))$. Finally, because $K_1(C_0(\dot{X}))$ sits in the middle of the exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}^m}{(1, \cdots, 1) \cdot \mathbb{Z}} \longrightarrow K_1(C_0(\dot{X})) \longrightarrow \mathbb{Z}^{2g} \longrightarrow 0,$$

it is free and finitely generated, hence isomorphic to $\mathbb{Z}^{2g+m-1}$. □

It now follows from Corollary 10 and from Bott periodicity that $K_0(C_0(T^*\dot{X})) \simeq \mathbb{Z}$ and $K_1(C_0(T^*\dot{X})) \simeq \mathbb{Z}^q$. From this, Corollary 12 and Proposition 21, we get:

Corollary 23. *$K_0(\mathfrak{A}/\mathfrak{K})$ and $K_1(\mathfrak{A}/\mathfrak{K})$ are both isomorphic to $\mathbb{Z}^{2g+m}$.*

Propositions 13 and 14 then imply:

Corollary 24. *$K_0(\mathfrak{A}) \simeq K_0(\mathcal{A}) \simeq \mathbb{Z}^{2g+m}$ and $K_1(\mathfrak{A}) \simeq K_1(\mathcal{A}) \simeq \mathbb{Z}^{2g+m-1}$.*

7. Composition Sequence

In this section we return to our initial assumption: X is an arbitrary compact manifold with boundary. We recall that $\mathcal{G}$ and $\mathfrak{G}$ were defined after (10). It is obvious that $\mathcal{G} = \ker \sigma$. Moreover, $\mathcal{G}$ contains all integral operators with smooth kernel; hence, $\mathfrak{K} \subset \mathfrak{G}$.

Lemma 25. *There exists a positive constant c, determined only by X, such that*

$$\inf_{G' \in \mathfrak{G}} ||A + G'|| \le c||\bar{\sigma}(A)||, \tag{43}$$

for every $A \in \mathfrak{A}$.

Proof: It is enough to prove (43) for $A = P_+ \in \mathfrak{A}_{11}$, with P denoting a classical zero-order pseudodifferential operator with the transmission property on X. Moreover, since $\mathfrak{K} \subset \mathfrak{G}_{11}$, it suffices to prove that $\inf_{C\in\mathfrak{K}} ||P_+ + C|| \le c||\sigma(P)||$.

We need to distinguish between the given P, regarded as a pseudodifferential operator on $\dot{X}$, and an extension $\tilde{P}$ of P to Ω. It follows from the classical norm estimate for pseudodifferential operators on closed manifolds (quoted between (16) and (17)) that, for any $\delta > 0$, there is a compact $\tilde{C}$ on $L^2(\Omega)$ such that $||\tilde{P} + \tilde{C}|| < (1+\delta)||\sigma(\tilde{P})||$. If C is the compact operator on $L^2(X)$ obtained from $\tilde{C}$ by truncation (i.e., C is the restriction to X, composed with $\tilde{C}$, composed with zero-extension), then it is obvious that $||P_+ + C|| \le ||\tilde{P} + \tilde{C}||$. Hence, the lemma will be proven if we show that the choice of $\tilde{P}$ can be made so that $||\sigma(\tilde{P})|| \le c||\sigma(P)||$, for some constant c depending only on X. But that follows from Seeley's extension [42], as in [38], Lemma 2.3. □

A necessary and sufficient condition for a polyhomogeneous pseudodifferential operator to have the transmission property was given by Boutet de Monvel [3] (see also [22], Theorem 18.2.15). For zero-order operators, his criterion says that a given P has the transmission property if and only if

$$\partial_x^\beta \partial_\xi^\alpha p_j(x', 0, 0, 1) = (-1)^{j+|\alpha|} \partial_x^\beta \partial_\xi^\alpha p_j(x', 0, 0, -1), \tag{44}$$

for all relevant x' and for all indices α, $\beta \ge 0$, and $j \le 0$, with p_j denoting the degree-j positively homogeneous component (smooth for $\xi \ne 0$) in the asymptotic expansion of the symbol of P with respect to coordinates for which the boundary is given by $x_n = 0$. This condition is invariant under coordinate changes that preserve the set $\{x_n = 0\}$, as can be proven using the standard rules of pseudodifferential calculus. Let us say that a given $p_0 \in C^\infty(S^*X)$ satisfies the transmission condition if its positively homogeneous extension to the cotangent bundle (smooth except on the zero section) satisfies (44) for $j = 0$. It is much easier to see (it takes only the chain rule) that this definition is also invariant under the appropriate coordinates changes. It is part of the content of Theorem 1 in [35], 2.3.3.1, that, if $p \in C^\infty(S^*X)$ satisfies the transmission condition, then there exists a pseudodifferential operator with the transmission property on X whose principal symbol is p.

Now we prove that condition (44) survives the norm closure, but only for $\alpha = \beta = 0$ and $j = 0$.

Theorem 5. *The kernel of $\bar{\sigma} : \mathfrak{A} \to C(S^*X)$ is equal to $\mathfrak{G}$. The image of $\bar{\sigma}$ consists of all those functions in $C(S^*X)$ which, over each point of the boundary, take the same value at the two covectors that vanish on the tangent space of ∂X.*

Proof: It follows immediately from Lemma 25 that $\ker \bar{\sigma} \subseteq \mathfrak{G}$. On the other hand, from $\mathcal{G} = \ker \sigma$, we get: $\mathfrak{G} \subseteq \overline{\ker \sigma} \subseteq \ker \bar{\sigma}$. This proves the first statement.

It follows from the Stone-Weierstrass Theorem that the set of all $p \in C^\infty(S^*X)$ satisfying the transmission condition (which is contained in the image of σ, by the preceding remarks) is dense in $C(S^*X/\sim)$, with $S^*X/\sim$ defined after (10). Since the image of $\bar{\sigma}$ is closed, that finishes the proof. □

Corollary 26. *The principal symbol homomorphism induces a C^*-algebra isomorphism between $\mathfrak{A}/\mathfrak{G}$ and $C(S^*X/\sim)$,*

The following generalization of Gohberg and Seeley's norm estimate follows immediately from Theorem 5 and the fact that every injective C*-algebra homomorphism is an isometry.

Corollary 27. *If P is a classical zero-order pseudodifferential operator with the transmission property on X, then $\inf_G ||P_+ + G|| = ||\sigma(P)||$, where the infimum is taken over all polyhomogeneous zero-order singular Green operators.*

The next theorem is not new (see [35], 2.3.4.4, Corollary 2), but it is perhaps appropriate to offer here this proof.

Theorem 6. *The boundary principal symbol induces an isomorphism from $\mathfrak{G}/\mathfrak{K}$ to $C(S^*\partial X) \otimes \mathfrak{K}_{\overline{\mathbb{R}}_+}$, where $\mathfrak{K}_{\overline{\mathbb{R}}_+}$ denotes the ideal of compact operators on $L^2(\overline{\mathbb{R}}_+)$.*

Proof: That $\bar{\gamma}$ defines an isometry $\mathfrak{G}/\mathfrak{K} \to C(S^*\partial X) \otimes \mathfrak{K}_{\overline{\mathbb{R}}_+}$ follows from (3), since $\sigma(P) = 0$ if P has negative order, and A_{jk} is compact, if $j \neq 1$ or $k \neq 1$, for all $A \in \mathfrak{A}$. To prove that this isometry is surjective, it is enough to show that $\operatorname{Im} \bar{\gamma}$ contains $C_0(S^*V, \mathfrak{K}_{\overline{\mathbb{R}}_+})$, for any V as in Lemma 4.

Given $f, g \in \mathcal{S}(\overline{\mathbb{R}}_+)$ and $p \in C_c^\infty(S^*V)$, let $\tilde{p}(x', \xi')$ denote the local expression of the homogeneous extension of p of degree (-1). Denoting also by f and g their extensions to $\overline{\mathbb{R}}_+$ vanishing on the negative half-axis, let φ and ψ denote the Fourier transforms of g and h, respectively, where $h(t) = f(-t)$. We then define $\tilde{g}(x', \xi', \xi_n, \eta_n) = \tilde{p}(x', \xi')\omega(|\xi'|)\varphi(\frac{\xi_n}{|\xi'|})\psi(\frac{\eta_n}{|\xi'|})$, with ω, as in Section 2, denoting an excision function.

It is straightforward to check (using [18], (1.2.38) and (2.3.25), for example) that $\tilde{g}$ is a singular Green symbol of order and class zero on the euclidean space. Moreover, it is polyhomogeneous and its homogeneous principal part (smooth for $\xi' \neq 0$) is given by $\tilde{g}_0(x', \xi', \xi_n, \eta_n) = \tilde{p}(x', \xi')\varphi(\frac{\xi_n}{|\xi'|})\psi(\frac{\eta_n}{|\xi'|})$. It then follows (from [18], (2.3.25) and (2.4.6), for example) that

$$\tilde{g}_0(x', \xi', D_n) = |\xi'|\, \tilde{p}(x', \xi') \cdot (\kappa_{|\xi'|} \circ |g\rangle \circ \langle f| \circ \kappa_{|\xi'|}^{-1}), \tag{45}$$

with $\circ$ denoting, above, composition of operators on $L^2(\overline{\mathbb{R}}_+)$.

Let G denote the pullback to X of $\rho_1 \tilde{G} \rho_1$, with $\tilde{G}$ denoting the singular Green operator of symbol $\tilde{g}$ and ρ_1 as in the proof of Lemma 4. It follows from (45) that $\gamma(G) = \iota(p \otimes (|g\rangle\langle f|))$, with ι denoting the Banach space isomorphism of $C_0(S^*V, \mathfrak{K}_{\overline{\mathbb{R}}_+})$ onto itself

$$F(x', \xi') \mapsto |\xi'| \cdot (\kappa_{|\xi'|} \circ F(x', \xi') \circ \kappa_{|\xi'|}^{-1}).$$

We are finished, because the set of all such $p \otimes (|g\rangle\langle f|)$ is dense. $\square$

From Theorems 5 and 6, we obtain the composition sequence $0 \subset \mathfrak{K} \subset \mathfrak{G} \subset \mathfrak{A}$, with $\mathfrak{A}/\mathfrak{G} \simeq C(S^*X/\sim)$ and $\mathfrak{G}/\mathfrak{K} \simeq C(S^*\partial X) \otimes \mathfrak{K}_{\overline{\mathbb{R}}_+}$, isomorphisms induced by the principal symbol and by the boundary principal symbol, respectively.

Acknowledgements

Conversations or email with many friends and colleagues had an influence, sometimes decisive, at various stages of this work. It is a pleasure for us to thank them all, and to name some of them: Severino Collier Coutinho, Ruy Exel, Robert Lauter, Sergiu Moroianu, Anton Savin, Thomas Schick, Mariusz Wodzicki.

This work was done while the first author was a guest of the Institute of Mathematics of the University of Potsdam. He wishes to express his gratitude to all members of the group Partial Differential Equations and Complex Analysis for their always gentle hospitality. His visit was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (Fapesp, Brazil, Proc. 00/00451-2), Deutsche Forschungsgemeinschaft (DFG, Germany), and the EU Research and Training Network 'Geometric Analysis' (Contract HPRN-CT-1999-00118).

References

[1] B. Blackadar, K-Theory for Operator Algebras, Cambridge University Press, 1998.

[2] J. B. Bost, *Principe d'Oka, K-théorie et systèmes dynamiques non commutatifs*; Invent. math. **101**, 261-333, 1990.

[3] L. Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une varieté à bord, I-II*; J. Anal. Math. **17**, 241-304, 1966.

[4] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*; Acta Math. **126**, 11-51, 1971.

[5] L. G. Brown, P. Green & M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C*-algebras*; Pacific J. Math. **71**, 349-363, 1977.

[6] L. A. Coburn, *The C*-algebra generated by an isometry I*; Bull. Amer. Math. Soc. **73**, 722-726, 1967.

[7] L. A. Coburn, *The C*-algebra generated by an isometry II*; Trans. Amer. Math. Soc. **137**, 211-217, 1969.

[8] H. O. Cordes, Elliptic Pseudo-Differential Operators - An Abstract Theory; Springer LNM **756**, 1979.

[9] J. Dixmier, C*-Algebras, North-Holland, 1977.

[10] R. G. Douglas, *On the spectrum of a class of Toeplitz operators*; J. Math. Mech. **18**, 433-436, 1968.

[11] R. G. Douglas, Banach Algebra Techniques in Operator Theory, Graduate Texts in Mathematics **179**, Springer, 1998.

[12] R. Exel, *A Fredholm operator approach to Morita equivalence*; K-Theory **7**-3, 285-308, 1993.

[13] B. V. Fedosov, *An analytic formula for the index of an elliptic boundary-value problem* I, II, III; Math. USSR Sbornik, **22**-1, 61-90, 1974, **24**-4, 511-535, 1974, **30**-3, 341-359, 1976.

[14] B. V. Fedosov, F. Golse, E. Leichtnam & E. Schrohe, *The noncommutative residue for manifolds with boundary*; J. Funct. Anal. **142**, 1-31, 1996.

[15] W. Fulton, Algebraic Topology, a First Course, Graduate Texts in Mathematics **153**, Springer, 1995.

[16] I. C. Gohberg, *On the theory of multidimensional singular integral operators*, Dokl. Akad. Nauk SSSR **133**, 1279-1282, 1960.

[17] B. Gramsch, *Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren*, Math. Ann. **269**, 27-71, 1984.

[18] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, Second Edition, Birkhäuser, 1996.

[19] G. Grubb & G. Geymonat, *The essential spectrum of elliptic systems of mixed order*; Math. Ann. **227**, 247-276, 1977.

[20] G. Grubb & E. Schrohe, *Trace expansions and the noncommutative residue for manifolds with boundary*; J. reine angew. Math. **536**, 167-207, 2001.
[21] L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*; Singular Integrals (Chicago, 1966), Proc. Symp. Pure Math. **10**, 138-183, 1967.
[22] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Springer, 1985.
[23] J. Kaminker, *Pseudo-differential operators and differential structures I*; Operator Algebras and K-theory (San Francisco, 1981), Contemporary Math. **10**, 99-127, Amer. Math. Soc., 1982.
[24] M. Karoubi, K-Theory - An Introduction, Springer, 1978.
[25] J. J. Kohn & L. Nirenberg, *An algebra of pseudo-differential operators*; Comm. Pure Appl. Math. **18**, 269-305, 1965.
[26] R. Lauter, Holomorphic functional calculus in several variables and Ψ^*-algebras of totally characteristic operators on manifolds with boundary, Thesis, Mainz, 1996.
[27] R. Lauter, B. Monthubert & V. Nistor, *Pseudodifferential Analysis on Continuous Family Grupoids*; Documenta Math. **5**, 625-655, 2000.
[28] R. Lauter & S. Moroianu, *Fredholm theory for degenerate pseudodifferential operators on manifolds with fibered boundaries*, Comm. PDE **26**(1& 2), 233-283, 2001.
[29] R. Melrose & V. Nistor, *K-Theory of C^*-algebras of b-pseudodifferential operators*; Geom. Funct. Anal. **8**, 88-122, 1998.
[30] B. Monthubert, Groupoïdes et calcul pseudo-différentiel sur le varietés à coins, Thesis, Paris 7, 1998.
[31] B. Monthubert, *Pseudodifferential calculus on manifolds with corners and groupoids*; Proc. Amer. Math. Soc. **127**-10, 2871-2881, 1999.
[32] R. Nest & B. Tsygan, *Algebraic index theorem*; Comm. Math. Phys. **172**-2, 223-262, 1995.
[33] R. Nest & E. Schrohe, *Dixmier's trace for boundary value problems*; Manuscripta Math. **96**, 203-218, 1998.
[34] R. Nest & E. Schrohe, *Hochschild cohomology of Boutet de Monvel's algebra*; in preparation.
[35] S. Rempel & B.-W. Schulze, Index Theory of Elliptic Boundary Problems, Akademie Verlag, 1982.
[36] S. Rempel & B.-W. Schulze, Parametrices and Boundary Symbolic Calculus for Elliptic Boundary Problems without the Transmission Property, Math. Nach. **105**, 45-149, 1982.
[37] M. Rørdam, F. Larsen & N. J. Laustsen, An Introduction to K-Theory for C*-Algebras, London Mathematical Society Student Texts **49**, Cambridge University Press, 2000.
[38] E. Schrohe, *Fréchet Algebra Techniques for Boundary Value Problems on Noncompact Manifolds: Fredholm Criteria and Functional Calculus via Spectral Invariance*; Math. Nach. **199**, 145-185, 1999.
[39] E. Schrohe, *A short introduction to Boutet de Monvel's calculus*; Approaches to singular analysis (Berlin, 1999), Oper. Theory Adv. Appl. **125**, 85-116, Birkhäuser, 2001.
[40] E. Schrohe & B. -W. Schulze, *Boundary Value Problems in Boutet de Monvel's Algebra for Manifolds with Conical Singularities I*, Pseudo-differential calculus and mathematical physics, Math. Top. **5**, 97-209, Akademie Verlag, 1994.
[41] B. -W. Schulze, *Topologies and Invertibility in Operator Spaces with Symbolic Structure*, Proceedings 9. TMP, Karl-Marx-Stadt, Teubner Texte zur Mathematik **111**, 257-270, 1989.
[42] R. T. Seeley, *Extensions of C^∞ functions defined in a half space*; Proc. Amer. Math. Soc. **15**, 625-626, 1964.
[43] R. T. Seeley, *Integro-differential operators on vector bundles*; Trans. Amer. Math. Soc. **117**, 167-204, 1965.
[44] L. Waelbroeck, Topological vector spaces and algebras, Springer LNM **230**, 1971.
[45] N. E. Wegge-Olsen, K-Theory and C*-Algebras, Oxford Science Publications, 1994.
[46] M. Wodzicki, Spectral asymmetry and noncommutative residue (in Russian), Thesis, Stekhlov Institute of Mathematics, Moscow, 1984.

2000 Mathematics Subject Classification: 58J40, 46L80 (58J32, 35S15, 19K56).

Instituto de Matemática e Estatística, Universidade de São Paulo,
Caixa Postal 66281, 05315-970 São Paulo, Brazil.
melo@ime.usp.br

Department of Mathematics, University of Copenhagen,

Universitetsparken 5, 2100 Copenhagen, Denmark.
rnest@math.ku.dk

Universität Potsdam, Institut für Mathematik,
Postfach 601553, 14415 Potsdam, Germany.
schrohe@math.uni-potsdam.de