Elliptic Theory on Manifolds with Edges. I[1]

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Differential Operators on Manifolds with Singularities

Analysis and Topology

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Contents

Chapter 6

Elliptic Theory on Manifolds with Edges

Preface

In this chapter, we shall acquaint the reader with analytic aspects of elliptic theory on manifolds with singularities of the simplest type, that is, edges. This theory was developed by Schulze (e.g., see (Schulze 1991), where one can find further references, and also (Egorov and Schulze 1997)).[1] Although the geometry of such manifolds is hardly more complicated than that of manifolds with isolated singularities, the construction of elliptic theory proves to be much more involved, and the choice of "right" spaces where the operators should act is by no means obvious. Hence in the first section we try to give a clear presentation of related definitions and constructions, emphasizing the motivations and omitting less important details (which will be filled in in subsequent sections). In particular, the reader will learn

- how edge-degenerate operators are defined and in which spaces they should be considered;
- what the notion of symbol looks like for such operators;
- why these operators, even under the ellipticity condition for the principal symbol, are Fredholm only in exceptional cases, and how to obtain well-posed problems for them;
- what similarities and differences exist between edge problems and boundary value problems;
- what notion of ellipticity is suitable for edge problems;
- how one states the finiteness theorem in this case.

Issued related to *pseudodifferential* operators are touched in the first section only as long as they are necessary for the understanding of statements and motivations. The theory of edge-degenerate pseudodifferential operators, which is not only the main tool in the proof of the finiteness theorem but also an important basis for the homotopies used in index theory (see subsequent chapters), will be developed in Section 2.[2] In the third section, by means of the theory of pseudodifferential operators, we construct regularizers and thus prove the finiteness theorem for elliptic edge problems (including the case of *pseudo*differential operators in the main equation of the problem).

The main body of the chapter deals with the so-called *conical* edges. In Section 4, we briefly discuss manifolds with cuspidal edges of arbitrary orders and explain what results remain valid in this case and how one can transfer the corresponding constructions.

The techniques used in this chapter involve some facts concerning operators in sections of infinite-dimensional bundles. For the case of zero-order operators, this material is mainly contained in Chapter 4. Additional information needed to consider operators of nonzero order is included in the Appendix to the present chapter.

[1] Note that operators on manifolds with edges are also studied in (Mazzeo 1991, Mazzeo and Melrose 1998).

[2] Sections 2–4 are not included in this preprint and will be published separately.

6.1. Motivation and Main Constructions

Just as in the chapter on manifolds with isolated singularities, we first of all describe the structure of the degenerate differential operators to be studied and the related geometry of singularities and then, using an analogy with boundary value problems, describe an approach to the statement of well-posed (i.e., Fredholm) problems for edge-degenerate elliptic operators on manifolds with edges and state ellipticity conditions and the finiteness theorem for edge problems.

6.1.1. Manifolds with edges

First, we introduce the main geometric object on which elliptic theory will be developed, namely, manifolds with edges.

Definition. Let M be a smooth locally compact manifold with boundary ∂M. Suppose that the boundary is equipped with the structure of a locally trivial bundle with base X and fiber Ω, where Ω and X are also smooth compact manifolds:

$$\pi : \partial M \xrightarrow{\Omega} X. \tag{6.1}$$

We identify all points in each fiber $\Omega_x = \pi^{-1}(x)$ of π with one another, i.e., introduce the following equivalence relation on M:

two points $a_1, a_2 \in M$ are equivalent ($a_1 \sim a_2$) if and only if either $a_1 = a_2$ or both points belong to ∂M and $\pi(a_1) = \pi(a_2)$. (6.2)

DEFINITION 6.1. The quotient space $\mathcal{M} = M/{\sim}$ with respect to the equivalence relation (6.2) is called a *manifold with edges* or, more precisely, a *manifold with edge X*.

The structure of a neighborhood of the edge. Let U be a collar neighborhood of the boundary in M. Once and for all, we choose a direct product structure

$$U \simeq \partial M \times [0, 1); \tag{6.3}$$

then the bundle (6.1) lifts to the bundle

$$\pi : U \longrightarrow X \times [0, 1) \tag{6.4}$$

(the projection acts as the identity operator with respect to the second argument). Passing to the quotient space, we see that the neighborhood $\mathcal{U} = U/{\sim}$ of the edge in $\mathcal{M}$ is also a bundle over X. The fiber of this bundle is a manifold with isolated singularity, namely, the neighborhood $\{r < 1\}$ of the vertex in the infinite cone

$$K_\Omega = (\Omega \times \mathbb{R}_+) \,/\, \bigl(\Omega \times \{0\}\bigr) \tag{6.5}$$

with base Ω.

Thus geometrically a manifold with edges is a space with singularities which looks like the direct product of a domain V in Euclidean space $\mathbb{R}^n$ by the cone K_Ω with smooth compact base Ω in a neighborhood of any singular point (see Fig. 6.1). The *edge* X (the set of singular points) itself is a smooth manifold and is locally represented by the product of V by the cone vertex.

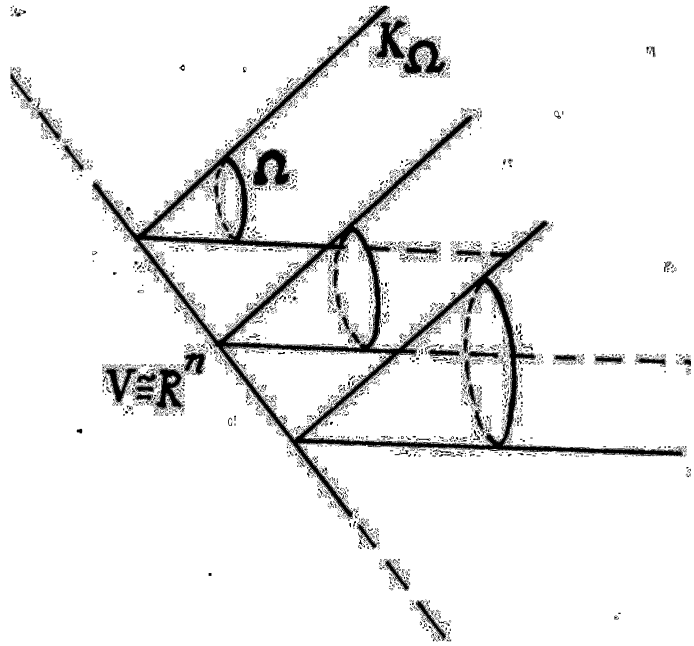

Figure 6.1. The local structure of a manifold with edges near the singularity

Next, there is a natural diffeomorphism

$$\mathcal{M} \setminus X \equiv \overset{\circ}{\mathcal{M}} \simeq \overset{\circ}{M} \equiv M \setminus \partial M.$$

Moreover, M can be reconstructed from $\mathcal{M}$ as the blow-up

$$\mathcal{M}^{\wedge} = M.$$

More precisely, each cone $K(x)$, $x \in X$, is blown up at the vertex.

◀ Details of the blow-up construction for cones can be found in the preceding chapter. ▶

Remark 6.2. Note that even though the manifold $\mathcal{M}$ with edges is the main geometric object of the theory and, in particular, it is this manifold that arises in examples and applications of the theory, it will be more convenient to carry out most of the main analytical constructions on the original manifold M with boundary.

Examples. Now let us give some simplest examples of manifolds with edges.

EXAMPLE 6.3. A manifold with boundary can always be viewed as a manifold with edges. Here the edge is the boundary, the projection

$$\pi : \partial M \longrightarrow X = \partial M$$

is the identity mapping, and the "cone" corresponding to each point of the boundary is just the half-interval $[0, 1)$. Thus, the base $\Omega = \{pt\}$ of the cone is a singleton. (Needless to say, to define the structure of a manifold with edges on M, one should choose a trivialization $U \simeq \partial M \times [0, 1)$ of a collar neighborhood of ∂M. One can do this, say, by choosing a Riemannian metric on M and by fibering U into geodesics normal to the boundary.)

EXAMPLE 6.4. We obtain a slightly more complicated example of a manifold with edge if we consider a manifold whose boundary is equipped with the structure of a covering. Let M be a manifold with boundary $\partial M = Y$, and let $Y \xrightarrow{\pi} X$ be an m-sheeted covering (a local diffeomorphism for which the preimage of each point consists of m elements). Then the cone K_Ω is the union of m half-intervals with common initial point and the manifold $\mathcal{M}$ with edge X obtained from (M, π) by the factorization indicated in Definition 6.1 looks like an "m-sheeted notebook" in a neighborhood of each boundary point (see Fig. 6.2).

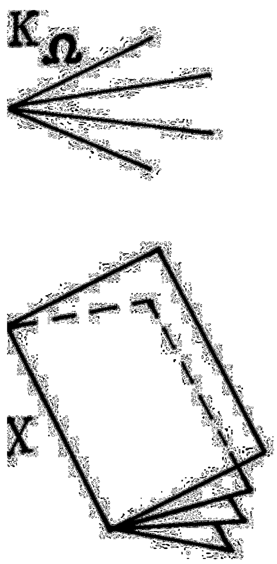

Figure 6.2. A cone K_Ω with finite base and a neighborhood of the singular point in the corresponding manifold with edges

EXAMPLE 6.5. Finally, let us describe a manifold with "internal" edges as an example. If $\mathcal{M}$ is a smooth manifold and $X \hookrightarrow \mathcal{M}$ is a smooth embedded submanifold, then one can equip $\mathcal{M}$ with the structure of a manifold with edge X as follows. Let us take some metric on $\mathcal{M}$; then the geodesic exponential map[3]

$$\exp : T\mathcal{M} \longrightarrow \mathcal{M}$$

[3]This map takes each vector $v \in T\mathcal{M}$ to the end of the geodesic of length $\|v\|$ with tangent vector $v/\|v\|$ at the initial point.

specifies an isomorphism of a sufficiently small neighborhood

$$V_\varepsilon = \{v \in NX \;\big|\; \|v\| < \varepsilon\}$$

of the zero section in the normal bundle $NX \hookrightarrow T\mathcal{M}$ of X onto a tubular neighborhood U of X in $\mathcal{M}$ and hence equips U with the structure of a bundle over X with fiber the ball, or, which is the same, the cone whose base is the sphere in NX.

EXERCISE 6.6. Check that the assumptions of Definition 6.1 are satisfied and the blow-up of the manifold $\mathcal{M}$ treated as a manifold with edge X is isomorphic to $\mathcal{M} \setminus U$.

◀ Note that the relation $\mathcal{M}^\wedge \simeq \mathcal{M} \setminus U$ holds for an arbitrary manifold with edges. ▶

A differential-geometric description. One can also describe a manifold with edges by differential-geometric means. To this end, one equips M with a Riemannian metric degenerating at the boundary in such a way that the distance between points lying in the same fiber is zero. We shall use a metric representable in U in the form

$$d\rho^2 = dr^2 + dx^2 + r^2 d\omega^2, \tag{6.6}$$

where $r \in [0,1)$ is the second coordinate in the decomposition (6.3) (one can readily see that this coordinate is just the distance to the boundary in the metric (6.6)) and dx^2 and $d\omega^2$ are some smooth nondegenerate metrics on X and Ω_x, respectively. (The latter metric is naturally assumed to depend smoothly on $x \in X$). In terms of the metric (6.6), the equivalence relation (6.2) can be described as follows:

$$a_1 \sim a_2 \text{ if and only if } \rho(a_1, a_2) = 0. \tag{6.7}$$

Remark 6.7. Needless to say, there are many other metrics satisfying (6.7). For example, the factor r^{2k} multiplying $d\omega^2$ specifies "cuspidal edges of order k" etc. We restrict ourselves to the case of the metric (6.6), and the corresponding edges are said to be *conical*. The case of cuspidal edges will be analyzed in § 4.

6.1.2. Edge-degenerate differential operators

Definition. What differential operators will be studied on manifolds with edges? In the interior $\overset{\circ}{\mathcal{M}} = \mathcal{M} \setminus X$ of the manifold, they will be just arbitrary differential operators with smooth coefficients, but how do they behave near the edge? To define a natural class of such operators, we use the metric. Namely, let $x = (x_1, \dots, x_n)$ and $\omega = (\omega_1, \dots, \omega_k)$ be local coordinates on X and Ω, respectively. Each of the vector fields

$$A_j = -i\frac{\partial}{\partial x_j}, \quad j = 1, \dots, n, \quad B = -i\frac{\partial}{\partial r}, \quad C_j = -i\frac{1}{r}\frac{\partial}{\partial \omega_j}, \quad j = 1, \dots, k, \tag{6.8}$$

has a length of the order of unity (uniformly bounded above and below by positive constants) in the metric (6.6). We intend to define our differential operators as polynomials of such vector fields with coefficients being smooth functions on M. Note that the operator tuple (6.8) is not closed with respect to commutators: the commutator

$$[B, C_j] = -i\frac{1}{r}C_j \tag{6.9}$$

cannot be expressed as an ordered polynomial[4] with smooth coefficients of the operators (6.8). Hence the set of ordered polynomials with smooth coefficients of the operators (6.8) is not closed with respect to composition: when moving the operator arguments in the product to the corresponding positions, we gain the singular factor $1/r$. To rectify the situation, we supplement our operator tuple with yet another operator

$$D = \frac{1}{r}. \tag{6.10}$$

DEFINITION 6.8. An *edge-degenerate differential operator of order* $\leq m$ on a manifold $\mathcal{M}$ with edge X is a differential operator P with smooth coefficients on $\overset{\circ}{M}$ such that in local coordinates (x, r, ω) in a neighborhood of the boundary it can be represented as a polynomial of the operators (6.8) and (6.10) with coefficients smooth up to $r = 0$:

$$P = \sum_{|\alpha|+|\beta|+j+l\leq m} a_{\alpha\beta jl}(x, \omega, r) A^\alpha B^j C^\beta D^l. \tag{6.11}$$

The set of such operators will be denoted by $\mathcal{D}_m$, and the set of edge-degenerate differential operators of all orders will be denoted by

$$\mathcal{D} = \bigcup_{m=0}^{\infty} \mathcal{D}_m. \tag{6.12}$$

We point out that the definition of the order of an operator near the edge counts not only differentiations but also the factors $1/r$.

Remark 6.9. Representability in the form (6.11) is independent of the choice of local coordinates.

PROPOSITION 6.10. *The set* $\mathcal{D}$ *of edge-degenerate differential operators is an algebra, and*

$$\mathcal{D}_m \mathcal{D}_{m'} \subseteq \mathcal{D}_{m+m'}. \tag{6.13}$$

(*In other words, the formula* (6.11) *specifies a filtration on* $\mathcal{D}$.)

◀ The proof is an easy exercise. ▶

An alternative representation of elements of $\mathcal{D}_m$**.** In our definition, edge-degenerate operators are expressed as polynomials of the arguments some of which have a singularity at $r = 0$ (i.e., near the boundary or the edge). In many cases (in particular, in the analysis of symbols in the next item) it is more convenient to use a different representation in which one extracts the singular factor so that the remainder proves to be a polynomial with smooth coefficients of vector fields *smooth up to the boundary*. (However, these fields degenerate at the boundary.) The following simple assertion holds.

PROPOSITION 6.11. *An edge-degenerate differential operator* $D \in \mathcal{D}_m$ *of order* m *on a manifold* $\mathcal{M}$ *with edges can be represented in local coordinates in a neighborhood of the edge in the form*

$$D = r^{-m} \sum_{|\alpha|+|\beta|+l\leq m} a_{\alpha\beta l}(x, \omega, r) \left(ir\frac{\partial}{\partial r}\right)^l \left(-ir\frac{\partial}{\partial x}\right)^\alpha \left(-i\frac{\partial}{\partial \omega}\right)^\beta, \tag{6.14}$$

where the coefficients $a_{\alpha\beta l}(x, \omega, r)$ *are smooth functions up to* $r = 0$.

EXERCISE 6.12. Prove this assertion.

[4] That is, as a polynomial in which the order of operator arguments is fixed; say, the A_j act first, B acts second, and the C_j act third; see (Maslov 1973) and also (Nazaikinskii, Sternin and Shatalov 1995) for a systematic exposition of the noncommutative operator calculus.

Example: passage to polar coordinates. Let us now give an example of an edge-degenerate operator. Just as an "artificial" singular point was obtained in the preceding chapter by the passage to polar coordinates on the plane, one can obtain an "artificial" edge by passing to polar coordinates with respect to *part* of the variables in a space of higher dimension. (This is of course a special case of the construction given in Example 6.5.) Consider, say, the space $\mathbb{R}^3$ with coordinates (x, y, z) and pass to polar coordinates (r, φ) in the (x, y)-plane. Then we obtain the decomposition

$$\mathbb{R}^3 = \mathbb{R} \times K$$

into the direct product of the line $\mathbb{R}$ with coordinate z by the cone K whose base is a circle with coordinate φ and whose generators are rays issuing from the origin in the (x, y)-plane. The coordinate on each of the rays is the polar radius r, and the "edge" arising in this construction is the line $x = y = 0$, i.e., the z-axis, so that z is the coordinate along the edge.

The Laplace operator

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in $\mathbb{R}^3$ can be rewritten in the new coordinates (r, φ, z) in the form

$$\triangle = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \varphi^2}\right] + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \varphi^2} + \left(r\frac{\partial}{\partial z}\right)^2\right]. \tag{6.15}$$

EXERCISE 6.13. Check that the operator (6.15) is the Beltrami–Laplace operator associated with the metric $d\rho^2 = dr^2 + dz^2 + r^2\, d\varphi^2$.

◀ *Hint*. The metric $d\rho^2$ is just the standard Euclidean metric rewritten in the polar coordinates. ▶

Thus the Laplace operator in polar coordinates is edge-degenerate.

What happens if we add lower-order terms to the Laplace operator, i.e., consider, say, the operator

$$D = \triangle + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d?$$

An easy computation shows that the expression of the new operator in the coordinates (r, φ, z) is still the product of r^{-2} by a polynomial of the operators $ir\partial/\partial r$, $-ir\partial/\partial z$, and $i\partial/\partial\varphi$:

$$D = r^{-2}L\left(ir\frac{\partial}{\partial r}, -i\frac{\partial}{\partial \varphi}, -ir\frac{\partial}{\partial z}\right), \quad \text{where}$$

$$L(p, q, \eta) = L_0(p, q, \eta) - ir(a\cos\varphi + b\sin\varphi)p + ir(b\cos\varphi - a\sin\varphi)q + irc\eta + r^2 d.$$

The coefficients of this polynomial are no longer constant. (However, the fact that the original operator has constant coefficients is essentially occasional.) It is more noteworthy that *the coefficients of first-order terms in this polynomial are divisible by r and the coefficients of zero-order terms are divisible by r^2*. This is however due to the fact that the operator in question has been obtained by the polar change of variables from an operator whose coefficients have no singularities at $r = 0$. In the general case, where the edge is real rather than fictitious (for example, the base of the cone is not a sphere of any dimension), the very notion of an operator without singularities is meaningless (for there are no original smooth coordinates), and it makes no sense to impose such divisibility conditions on the coefficients.

Function spaces. Now we construct a natural scale of spaces in which edge-degenerate differential operators act. This scale is constructed by analogy with the Sobolev scale (and coincides with it in the interior of the manifold).

Recall that the Sobolev scale on a smooth compact manifold M is constructed as follows. As the "reference" (zero) space of the scale, one takes the space $L^2(M)$ (with respect to an arbitrary Riemannian volume form), and then the spaces $H^s(M)$ are defined as the domains of the respective operators $(1-\triangle)^{s/2}$, where $\triangle$ is the Beltrami–Laplace operator on M, for positive s and as the dual spaces of $H^{-s}(M)$ with respect to the pairing given by the inner product in $L^2(M)$ for negative s. The Beltrami–Laplace operator is locally equivalent to the sum of squares of coordinate vector fields, and so this definition provides the continuity of differential (and pseudodifferential) operators in the scale $H^s(M)$ with order equal to the order of the operator.

Let us return to the case of manifolds with edges. Here the natural candidate for the reference space of the scale is $L^2(\mathcal{M}, d\mathrm{vol})$, where $d$vol is the Riemannian volume form corresponding to the metric $d\rho^2$. Near the edge this form is given by

$$d\mathrm{vol} = r^k dr d\omega dx, \quad k = \dim \Omega, \tag{6.16}$$

where $d\omega$ and dx are the Riemannian volume forms on Ω_x and X, respectively. Locally, it is natural to take the operator generating the scale in the form

$$\widetilde{T} = 1 + A^*A + B^*B + C^*C + D^*D,$$

where the asterisk stands for the adjoint operator with respect to the inner product in $L^2(\mathcal{M}, d\mathrm{vol})$. One can readily see that $\widetilde{T}$ is equivalent to the globally defined operator

$$T = 1 - \triangle_M + \frac{1}{r^2}, \tag{6.17}$$

where $\triangle_M$ is the Beltrami–Laplace operator on M associated with the metric $d\rho^2$ and r is the distance to the boundary, which is globally well defined on M. One can show that T is essentially self-adjoint on $C_0^\infty(\overset{\circ}{M})$ in $L^2(\mathcal{M}, d\mathrm{vol})$. Hence the closure of T (which we denote by the same letter) is a positive self-adjoint operator, and we define a scale of spaces $\mathcal{W}^s(\mathcal{M})$, $s \in \mathbb{R}$, as follows.

DEFINITION 6.14. For $s \geq 0$, the space $\mathcal{W}^s(\mathcal{M})$ coincides with the domain of the operator $T^{s/2}$ and is equipped with the inner product

$$(u, v)_s = (T^{s/2}u, T^{s/2}v)_0, \tag{6.18}$$

where $(\,\cdot\,, \,\cdot\,)_0$ is the inner product in $L^2(\mathcal{M}, d\mathrm{vol})$. For $s < 0$, the space $\mathcal{W}^s(\mathcal{M})$ is the dual of $\mathcal{W}^{-s}(\mathcal{M})$ with respect to the inner product in $L^2(\mathcal{M}, d\mathrm{vol})$.

The following theorem can be proved by standard techniques of functional analysis.

THEOREM 6.15. *The family $\mathcal{W}^s(\mathcal{M})$, $s \in \mathbb{R}$, is a scale of Hilbert spaces. Outside an arbitrarily small neighborhood of the wedge, the space $\mathcal{W}^s(\mathcal{M})$ coincides with the corresponding Sobolev space $H^s(\mathcal{M})$. The linear manifold $C_0^\infty(\overset{\circ}{M})$ is dense in all spaces of this scale. An arbitrary edge-degenerate operator $P \in \mathcal{D}_m$ of order m is continuous with step m in this scale*:

$$P : \mathcal{W}^s(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m}(\mathcal{M}). \tag{6.19}$$

6.1.3. Symbols

Definition. Let us find out what the notion of *principal symbol* looks like for edge-degenerate operators. Recall that for the case of a smooth manifold the space of mth-order principal symbols, i.e., homogeneous polynomials of degree m with respect to the momentum variables is naturally identified with the quotient space of operators of order m by operators of order $m-1$: the symbol is uniquely determined by the principal part of the operator and in turn uniquely determines the operator modulo lower-order terms. By analogy with this, we give the definition of the principal symbol in our case. We know that the algebra $\mathcal{D}$ of edge-degenerate differential operators is filtered by the order of the operator, and we define symbols as elements of the associated graded algebra

$$\operatorname{gr}\mathcal{D} = \bigoplus_{m=0}^{\infty} \mathcal{D}_m/\mathcal{D}_{m-1} \qquad (\mathcal{D}_{-1} \stackrel{\text{def}}{=} \mathbf{0}).$$

DEFINITION 6.16. The *principal symbol* (of order m) of an edge-degenerate differential operator $P \in \mathcal{D}_m$ of order $\leq m$ is the image of P in the quotient space $\mathcal{D}_m/\mathcal{D}_{m-1}$ under the natural projection:

$$\mathcal{D}_m \ni P \longmapsto \Sigma(P) \in \mathcal{D}_m/\mathcal{D}_{m-1}. \tag{6.20}$$

Computation. To use this definition in practice, we however need to describe the structure of the principal symbol in more detail. Thus let us find the mth gradation component $[\operatorname{gr}\mathcal{D}]_m = \mathcal{D}_m/\mathcal{D}_{m-1}$. We make the computations in coordinates. First, consider a coordinate neighborhood W that does not contain points of the edge. Let $y = (y_1, \dots, y_{m+k+1})$ be local coordinates in this neighborhood. The elements of $\mathcal{D}_m$ are differential operators with smooth coefficients of the form

$$D = \sum_{|\alpha|=1}^{m} a_\alpha(y)\left(-i\frac{\partial}{\partial y}\right)^\alpha,$$

while elements of $\mathcal{D}_{m-1}$ have the same form except that summation stops at $m-1$ rather than m. Hence elements of the quotient space $[\operatorname{gr}\mathcal{D}]_m$ can be represented in this neighborhood by homogeneous differential operators of the form

$$\widetilde{D} = \sum_{|\alpha|=m} a_\alpha(y)\left(-i\frac{\partial}{\partial y}\right)^\alpha. \tag{6.21}$$

(In other words, the operator $\widetilde{D}$ is obtained from D by the omission of lower-order terms.) Is this representation satisfactory? Not really, since

(1) it is coordinate-dependent (under changes of coordinates in the operator (6.21), the homogeneity is lost);
(2) the product of two homogeneous operators is not homogeneous of the total degree (the composition formula involves lower-order terms with derivatives of the coefficients), so this representation is not consistent with the multiplication rule in $\operatorname{gr}\mathcal{D}$.

How can we rectify this? The remedy is simple: in (6.21), we replace the differentiations $-i\partial/\partial y$ by variables $\theta = (\theta_1, \dots, \theta_{m+n+1})$ commuting with each other and with y, i.e., proceed from elements of the form (6.21) to homogeneous polynomials

$$\sigma(D) = \sum_{|\alpha|=m} a_\alpha(y)\theta^\alpha. \tag{6.22}$$

The homogeneous polynomial (6.22) is just the *principal symbol* of the operator D. Now everything is okay: the symbol of a product of operators is equal to the product of symbols of the factors, and the principal symbol is transformed under changes of coordinates by the well known rule (as a function on the cotangent bundle of the manifold).

So far we have analyzed only the smooth part of the manifold and said essentially nothing new compared with the classical case. Now let us carry out a similar argument in a neighborhood of a point of the edge in the above-mentioned local coordinates (x, ω, r).

Elements D of the space $\mathcal{D}_m$ have the form (6.14), and elements d of the subspace $\mathcal{D}_{m-1} \subset \mathcal{D}_m$ have the same form with m replaced by $m-1$:

$$d = r^{-m+1} \sum_{|\alpha|+|\beta|+l\le m-1} b_{\alpha\beta l}(x,\omega,r)\left(ir\frac{\partial}{\partial r}\right)^l \left(-ir\frac{\partial}{\partial x}\right)^\alpha \left(-i\frac{\partial}{\partial \omega}\right)^\beta. \tag{6.23}$$

Multiplying and dividing (6.23) by r, we obtain a representation of d with the same factor r^{-m} multiplying the sum as in (6.14), which is more convenient for our argument:

$$d = r^{-m} \sum_{|\alpha|+|\beta|+l\le m-1} r b_{\alpha\beta l}(x,\omega,r)\left(ir\frac{\partial}{\partial r}\right)^l \left(-ir\frac{\partial}{\partial x}\right)^\alpha \left(-i\frac{\partial}{\partial \omega}\right)^\beta. \tag{6.24}$$

Comparing (6.14) with (6.24), we see that elements of the subspace $\mathcal{D}_{m-1} \subset \mathcal{D}_m$ are singled out in $\mathcal{D}_m$ by the following conditions: (a) the sum contains no terms of order m; (b) all coefficients are divisible in the class of smooth functions by r. Hence the elements of the quotient space $[\operatorname{gr}\mathcal{D}]_m$ can be represented in this neighborhood by differential operators of the form

$$\begin{aligned}\widetilde{D} = r^{-m} &\sum_{|\alpha|+|\beta|+l=m} a_{\alpha\beta l}(x,\omega,r)\left(ir\frac{\partial}{\partial r}\right)^l \left(-ir\frac{\partial}{\partial x}\right)^\alpha \left(-i\frac{\partial}{\partial \omega}\right)^\beta \\ &+ r^{-m} \sum_{|\alpha|+|\beta|+l\le m-1} \widetilde{a}_{\alpha\beta l}(x,\omega)\left(ir\frac{\partial}{\partial r}\right)^l \left(-ir\frac{\partial}{\partial x}\right)^\alpha \left(-i\frac{\partial}{\partial \omega}\right)^\beta. \end{aligned} \tag{6.25}$$

Here

$$\widetilde{a}_{\alpha\beta l}(x,\omega) = a_{\alpha\beta l}(x,\omega,0); \tag{6.26}$$

i.e., $\widetilde{D}$ is obtained from D by the replacement of all coefficients of lower-order terms in the sum by their values at $r=0$.

EXERCISE 6.17. Verify that $D - \widetilde{D} \in \mathcal{D}_{m-1}$ and that $\widetilde{D} = 0$ if and only if $D \in \mathcal{D}_{m-1}$.

Is this representation completely satisfactory? It is not, for the same reason as (6.21): it does not properly behave under changes of variables and does not respect the multiplication law in $\operatorname{gr}\mathcal{D}$. How to fix this representation? Let us analyze the expression (6.25) more thoroughly. It essentially comprises two components of rather different nature, the higher-degree component *containing the values of* *the coefficients for all* r and the lower-degree component that *depends only on the values of the coefficients at* $r = 0$. We already know how to rectify the first component: we should transform it into the principal symbol by replacing the differentiations with commuting variables, which ensures the desired multiplicativity and invariance under changes of variables. Now we should rectify the second component, but a brief analysis shows that this is impossible in principle, since under changes of variables the second component inherits some commutator terms from the first component. There is only one way: we should sacrifice the independence of the components and supplement the second component with some information pertaining to the first component. Fortunately, one can easily do this in a natural way. The first component involves the values of the coefficients for all r, hence in particular for $r = 0$! We *add* the first component with coefficients freezed at $r = 0$ to the second. In other words, instead of one operator $\widetilde{D}$ representing D in $\operatorname{gr}\mathcal{D}$, we consider the pair of operators

$$\widetilde{D}_m = r^{-m} \sum_{|\alpha|+|\beta|+l=m} a_{\alpha\beta l}(x,\omega,r)\left(ir\frac{\partial}{\partial r}\right)^l\left(-ir\frac{\partial}{\partial x}\right)^\alpha\left(-i\frac{\partial}{\partial\omega}\right)^\beta, \tag{6.27}$$

$$\widetilde{D}_\wedge = r^{-m} \sum_{|\alpha|+|\beta|+l\le m} \widetilde{a}_{\alpha\beta l}(x,\omega)\left(ir\frac{\partial}{\partial r}\right)^l\left(-ir\frac{\partial}{\partial x}\right)^\alpha\left(-i\frac{\partial}{\partial\omega}\right)^\beta. \tag{6.28}$$

The first operator is obtained from D by the omission of all lower-order terms, and the second operator, by setting $r = 0$ in all coefficients $a_{\alpha\beta l}$. These operators are not independent: the compatibility condition (6.26) must hold for $|\alpha| + |\beta| + l = m$. Now we make the operators (6.27) and (6.28) into objects such that not only linearity but also multiplicativity and invariance under changes of variables take place. To this end, in the operator (6.27) we replace the operator arguments by variables commuting with each other and with the arguments of the coefficients by the rule

$$ir\frac{\partial}{\partial r} \longmapsto p, \quad -ir\frac{\partial}{\partial x} \longmapsto \eta, \quad -i\frac{\partial}{\partial\omega} \longmapsto q$$

and drop the factor r^{-m}; in the operator (6.28), we replace only one operator argument by the rule

$$-i\frac{\partial}{\partial x} \longmapsto \xi.$$

As a result, we obtain the function

$$\sigma(D)(x,\omega,r,\eta,q,p) = \sum_{|\alpha|+|\beta|+l=m} a_{\alpha\beta l}(x,\omega,r)p^l\eta^\alpha q^\beta \tag{6.29}$$

and the family

$$\sigma_\wedge(D)(x,\xi) = r^{-m} \sum_{|\alpha|+|\beta|+l\le m} \widetilde{a}_{\alpha\beta l}(x,\omega)\left(ir\frac{\partial}{\partial r}\right)^l (r\xi)^\alpha\left(-i\frac{\partial}{\partial\omega}\right)^\beta \tag{6.30}$$

of cone-degenerate operators with parameters (x,ξ)

PROPOSITION 6.18. (1) *Under changes of local coordinates on* ∂M *respecting the bundle structure* $\pi : \partial M \longrightarrow X$*, the function* (6.29) *associated with a given edge-degenerate operator* D *of order* m *on*

$\mathcal{M}$ is transformed as a function on the cotangent bundle $T^*\partial M$ depending on the additional parameters r and p.

(2) *For any given (x,ξ), the operator* (6.30) *associated with D is a well defined differential operator on the cone K_Ω (i.e., is independent of the choice of local coordinates on Ω). Moreover, under changes of variables on X the family* (6.30) *behaves as an operator-valued function on T^*X.*

(3) *The equations $\sigma(D) = 0$ and $\sigma_\wedge(D) = 0$ hold if and only if $D \in \mathcal{D}_{m-1}$.*

(4) *The correspondence $D \longmapsto (\sigma(D), \sigma_\wedge(D))$ is linear and multiplicative. This means that*

$$\sigma(D_1 D_2) = \sigma(D_1)\sigma(D_2), \quad \sigma_\wedge(D_1 D_2) = \sigma_\wedge(D_1)\sigma_\wedge(D_2).$$

EXERCISE 6.19. Check the validity of these assertions.

DEFINITION 6.20. The function $\sigma(D)$ is called the *principal symbol* and the family $\sigma_\wedge(D)$ is called the *edge symbol* of the operator D.

The cotangent bundle and the principal symbol. So far, we have essentially defined *two* principal symbols, one given by the formula (6.29) in a neighborhood of the singularity set and the other, traditional principal symbol (6.22) used in the interior part of the manifold. How do they agree with each other? It turns out that one can naturally define the cotangent bundle of a manifold with edges in such a way that the principal symbol proves to be a function on its total space and the formulas (6.22) and (6.29) just give the local expressions of this function in special coordinate systems. Let us present the corresponding construction (which can be omitted in the first reading).

This construction is a slight modification of the "compressed cotangent bundle" (Melrose 1981) (see also Chap. 5) Here we again start from the manifold $M = \mathcal{M}^\wedge$ with boundary, which is the blow-up of the manifold with edges which we intend to describe. Recall that the boundary $Y = \partial M$ is equipped with the structure of a locally trivial bundle

$$\pi : Y \longrightarrow X$$

with smooth base X and smooth fiber Ω. Let $\mathcal{V}$ be the Lie algebra of smooth vector fields on M tangent to the fibers of π everywhere on the boundary. This algebra describes the manifold $\mathcal{M}$ with a conical edge obtained by shrinking the fibers of π into points. Consider the space $\mathcal{Z}$ of vector fields v on $\overset{\circ}{M}$ such that $rv \in \mathcal{V}$ (that is, more precisely, rv extends to a smooth vector field on M and that field belongs to $\mathcal{V}$).

The space $\mathcal{Z}$ (as well as $\mathcal{V}$) is a $C^\infty(M)$-module. Consider the dual module $\mathrm{Hom}_{C^\infty(M)}(\mathcal{Z}, C^\infty(M))$. (Its elements are $C^\infty(M)$-linear mappings of the space $\mathcal{Z}$ into $C^\infty(M)$; they can be added and multiplied by smooth functions in a natural way.) Both $\mathcal{Z}$ and $\mathrm{Hom}_{C^\infty(M)}(\mathcal{Z}, C^\infty(M))$ are locally free $C^\infty(M)$-modules, and by the Serre–Swan theorem (e.g., see (Atiyah 1989)) they are the modules of smooth sections of some vector bundles over M. These bundles are called the *tangent bundle* and the *cotangent bundle* of $\mathcal{M}$, respectively.

DEFINITION 6.21. The *tangent bundle* of the manifold $\mathcal{M}$ is the (unique) vector bundle $T\mathcal{M}$ over M such that

$$C^\infty(M, T\mathcal{M}) = \mathcal{Z}. \tag{6.31}$$

The *cotangent bundle* of the manifold $\mathcal{M}$ is the (unique) vector bundle $T^*\mathcal{M}$ over M such that

$$C^\infty(M, T^*\mathcal{M}) = \mathrm{Hom}_{C^\infty(M)}(\mathcal{Z}, C^\infty(M)). \tag{6.32}$$

The cotangent bundle is a smooth manifold $T^*\mathcal{M}$ with boundary. It is (noncanonically) isomorphic to T^*M. Since in a neighborhood of an arbitrary point $a \in \overset{\circ}{M}$ the space $\mathcal{Z}$ coincides with the space of all smooth vector fields, it follows from the definition that there is a (natural) isomorphism of the interiors

$$T^*\mathcal{M} \setminus \partial T^*\mathcal{M} \equiv T^*\overset{\circ}{\mathcal{M}} \quad \text{and} \quad T^*M \setminus \partial T^*M \equiv T^*\overset{\circ}{M}$$

of these manifolds. Similar assertions are valid for the tangent bundle. Note that there is a natural nondegenerate pairing

$$\langle \cdot , \cdot \rangle : T_a\mathcal{M} \times T_a^*\mathcal{M}, \quad a \in M, \tag{6.33}$$

compatible with the isomorphisms (6.31) and (6.32):

$$\phi(v)(a) = \langle v(a), \phi(a)\rangle, \quad v \in \mathcal{Z}, \quad \phi \in \operatorname{Hom}_{C^\infty(M)}(\mathcal{Z}, C^\infty(M)), \quad a \in M. \tag{6.34}$$

Now we can define the principal symbol of an edge-degenerate operator $D \in \mathcal{D}_m$ as a function on $T^*\mathcal{M}$. This can be done as follows. First, an arbitrary vector field $v \in \mathcal{Z}$ defines a smooth function on $T^*\mathcal{M}$ linear in the fibers by the formula

$$v(\rho) = \langle v(a), \rho\rangle, \quad \rho \in T_a^*\mathcal{M}. \tag{6.35}$$

The operator D can be represented in the form

$$D = \sum_{k=1}^{m} \sum_{j_1,\dots,j_k} b_{j_1,\dots,j_k} v_{j_1} \cdots v_{j_k}, \tag{6.36}$$

where $v_{j_s} \in \mathcal{Z}$ and all coefficients $b_{j_1,\dots,j_m}$ with multiindex of length m are smooth functions on M. (Lower-order terms may have power-law singularities on ∂M.) We define the principal symbol of D by the formula

$$\sigma(D)(\rho) = i^m \sum_{j_1,\dots,j_m} b_{j_1,\dots,j_m}(a) v_{j_1}(\rho) \cdots v_{j_k}(\rho), \quad \rho \in T_a^*\mathcal{M}. \tag{6.37}$$

EXERCISE 6.22. Prove that this is independent of the particular choice of the representation (6.36).

Now we see that formulas (6.22) and (6.29) just give the expressions of the principal symbol (6.37) in special coordinate systems on $T^*\mathcal{M}$. In the first case, this is the coordinate system (y, θ), where $\theta = (\theta_1, \dots, \theta_{n+k+1})$ are the fiber coordinates dual to the basis $(\partial/\partial y_1, \dots, \partial/\partial y_{n+k+1})$, and in the second case, the coordinate system $(x, \omega, r, \eta, q, p)$, where (η, q, p) are the fiber coordinates dual to the basis $(\partial/\partial x, r^{-1}\partial/\partial\omega, \partial/\partial r)$.

EXERCISE 6.23. Carry out the corresponding computations.

The compatibility condition. The symbols (6.29) and (6.30) are not independent but satisfy a compatibility condition inherited from the operators (6.27) and (6.28). To state this condition, consider the principal symbol of the family (6.30) viewed as an operator of order m with parameter ξ (and with the additional parameter x). This symbol has the form

$$\sigma(\sigma_\wedge(D))(x, \omega, r, \xi, q, p) = r^{-m} \sum_{|\alpha|+|\beta|+l=m} \widetilde{a}_{\alpha\beta l}(x, \omega) p^l (r\xi)^\alpha q^\beta. \tag{6.38}$$

(We still use the correspondence $ir\partial/\partial r \longmapsto p$, $-i\partial/\partial\omega \longmapsto q$.) Next, let

$$\sigma_\partial(D)(x, \omega, \eta, q, p) = \sigma(D)\,\big|_{r=0} \tag{6.39}$$

be the restriction of the principal symbol (6.29) to the set $r = 0$ (i.e., to the boundary of the cotangent bundle.) Then the compatibility condition has the form

$$r^m \sigma(\sigma_\wedge(D))(x, \omega, r, \xi, q, p) = \sigma_\partial(D)(x, \omega, r\xi, q, p). \tag{6.40}$$

This condition is obviously necessary and sufficient for the existence of a differential operator with given edge and principal symbols.

We can summarize this as follows:

The homogeneous component of degree m of the graded algebra $\operatorname{gr}\mathcal{D}$ associated with the algebra $\mathcal{D}$ of edge-degenerate differential operators on a manifold M with edges consists of pairs (*principal symbol, edge symbol*) *satisfying the compatibility condition* (6.40). *These pairs can naturally be viewed as symbols of such differential operators.*

Homogeneity properties of symbols. In the case of differential operators on smooth manifolds, the principal symbol $\sigma(D)$ of an operator of order m is a polynomial of degree m and hence is homogeneous:

$$\sigma(D)(y,\lambda\eta) = \lambda^m \sigma(D)(y,\eta), \quad \lambda > 0.$$

The same is true for the principal symbol of an edge-degenerate differential operator. Does the edge symbol have some homogeneity property?

The answer is "yes," but the property is more complicated. Note that the parameter ξ occurs in the edge symbol $\sigma_\wedge(D)$ of an edge-degenerate operator of order m only in the form of the product $r\xi$ raised to various powers. If we multiply ξ by $\lambda > 0$ and simultaneously divide r by λ, then the product remains intact. How does the division of r by λ affect the edge symbol? The variable r occurs in the edge symbol in the form of the common factor r^{-m} (which is multiplied by λ^m) and in the combination $r\partial/\partial r$, which is invariant under scalings of r. Thus if we multiply ξ by λ and simultaneously divide r by λ, then the operator is just multiplied by λ^m. How does one "divide r by λ" in an *operator*? This is actually quite easy to comprehend. Operators act on functions. To divide r by λ in an operator, one multiplies the argument r of the function by λ, applies the operator, and then divides the argument r of the resulting function by λ.

Let $\varkappa_\lambda$ be the operator multiplying the argument r of a function by λ:

$$\varkappa_\lambda u(r) = u(\lambda r). \tag{6.41}$$

(The function u may well have other arguments, which remain unaffected.) Then the desired homogeneity property of the edge symbol can be written in the form

$$\sigma_\wedge(D)(x,\lambda\xi) = \lambda^m \varkappa_\lambda \sigma(D)(x,\xi)\varkappa_\lambda^{-1}, \quad \lambda \in \mathbb{R}_+. \tag{6.42}$$

This property is referred to as *twisted homogeneity*.

EXERCISE 6.24. Check the formula (6.42).

Remark 6.25. The family $\varkappa_\lambda$ of scaling operators is a multiplicative one-parameter group of linear operators: $\varkappa_{\lambda\mu} = \varkappa_\lambda\varkappa_\mu$, $\lambda,\mu \in \mathbb{R}_+$.

Remark 6.26. In what follows, instead of (6.41) we use a modified definition of the group $\varkappa_\lambda$, which differs from (6.41) by a scalar factor:

$$\varkappa_\lambda u(r) = \lambda^{(k+1)/2} u(\lambda r), \tag{6.43}$$

where $k = \dim\Omega$ is the dimension of the base of the cone for the considered manifold with edges. This modification does not affect the property (6.42), since the scalar factor $\lambda^{(k+1)/2}$ commutes with the edge symbol, but the group (6.41) proves to be unitary in the main spaces of the scales introduced here (say, in $\mathcal{W}^0(W)$, where W is the infinite wedge serving as a model of $\mathcal{M}$ near the edge).

6.1.4. Elliptic problems

Statement of the problem. We say that an edge-degenerate operator D on a manifold $\mathcal{M}$ with edges is *formally elliptic* if its principal symbol is everywhere invertible on the cotangent bundle $T^*\mathcal{M}$ outside the zero section. We wish to study an equation of the form

$$Du = f \tag{6.44}$$

on the manifold M. The main question of interest to us is as follows:

- In what function spaces is it natural to consider this equation, and should one subject the solution u and the right-hand side f to additional conditions to make the equation uniquely solvable (or at least Fredholm)? What is the form of these conditions?

Reminder: boundary value problems. To understand how to seek the answer to this question, recall the similar (but simpler in many respects) case of elliptic operators on a manifold with boundary.

EXAMPLE 6.27. Consider the following example. Suppose that we study the first boundary value problem (the Dirichlet problem)

$$Lu \equiv \frac{\partial^2 u}{\partial t^2} + \triangle_x u = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}^n,$$
$$u|_{t=0} = u_0$$

for the Laplace equation in the half-space $t > 0$. The standard method is as follows. We make the Fourier transform with respect to the variables x and obtain the family of equations

$$\widehat{L}(\xi)\widetilde{u} \equiv \frac{\partial^2 \widetilde{u}}{\partial t^2} - |\xi|^2 \widetilde{u} = 0 \tag{6.45}$$

with parameter ξ. The general solution of this family has the form

$$\widetilde{u}(\xi, t) = C_0(\xi)e^{-|\xi|t} + C_1(\xi)e^{|\xi|t}. \tag{6.46}$$

(The functions $C_j(\xi)$ are the half-sum and the half-difference of the Fourier transforms of the initial condition and its first t-derivative and generally decay as $|\xi| \to \infty$ at most as some negative power of $|\xi|$.)

In what spaces should one solve the family (6.45)? The solution $u(x)$ of the equation itself is usually sought in the Sobolev spaces $H^s(\mathbb{R}^{n+1}_+)$ (for sufficiently large s). After the Fourier transform from t to ξ, by virtue of the Parseval identity, for integer s we have

$$\|u\|^2_{H^s(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^n_\xi \times \mathbb{R}^t_+} \left| \left(1 + |\xi|^2 - \frac{\partial^2}{\partial t^2}\right)^{s/2} \widetilde{u}(\xi, t) \right|^2 d\xi dt = \int_{\mathbb{R}^n_\xi} \|\widetilde{u}(\xi, \cdot)\|^2_{s,\xi}\, d\xi,$$

where the norm

$$\|\widetilde{v}\|^2_{s,\xi} = \int_0^\infty \left| \left(1 + |\xi|^2 - \frac{\partial^2}{\partial t^2}\right)^{s/2} \widetilde{v}(t) \right|^2 dt \tag{6.47}$$

is equivalent *for each given* ξ to the norm

$$\|v\|_s^2 = \int_0^\infty \left| \left(1 - \frac{\partial^2}{\partial t^2}\right)^{s/2} v(t) \right|^2 dt$$

in the space $H^s(\mathbb{R}_+)$. Thus it is natural to seek solutions of the family of equations (6.45) for each given ξ in the space $H^s(\mathbb{R}_+)$ while watching that the integral with respect to ξ expressing the norm of the original function u be convergent.

To ensure that the function (6.46) be indeed the Fourier transform of the solution, we need that the second term on the right-hand side disappear. (In general, it has an exponential growth as $|\xi| \to \infty$ for $t > 0$.) Then the function $C_0(\xi)$ is just the Fourier transform of the boundary condition, which results in the theorem on the well-posedness of the Dirichlet problem.

Remark 6.28. The decay condition is usually stated not as the decay with respect to ξ but as the decay as $t \to \infty$: "Consider the space of solutions of the model equation decaying as $t \to \infty$." At first glance, this statement is hard to understand, since it is not clear why one should pay attention to large t in the model problem. (One disposes of large t anyway when gluing the boundary solution with the solution in the interior of the domain.) The secret is that the model operator possesses the special *twisted homogeneity*

$$\widehat{L}(\lambda\xi) = \lambda^2 \varkappa_\lambda L(\xi) \varkappa_\lambda^{-1}, \tag{6.48}$$

where $\varkappa_\lambda$ is the dilation group

$$\varkappa_\lambda f(t) = \lambda^{1/2} f(\lambda t), \tag{6.49}$$

so that in the space of solutions of the homogeneous equation there is a basis consisting of $\varkappa_\lambda$-invariant solutions

$$|\xi|^{1/2} e^{\pm|\xi| t},$$

and for such solutions the growth (or decay) as $t \to \infty$ for a given $\xi \neq 0$ and as $|\xi| \to \infty$ for a given $t \neq 0$ are equivalent!

Remark 6.29. Note in passing that the ξ-dependent norm (6.47) also satisfies the *homogeneity condition*

$$\left\| \varkappa_{[\xi]} v \right\|_{s,\xi} = [\xi]^s \left\| v \right\|_s, \tag{6.50}$$

where

$$[\xi] = (1 + |\xi|^2)^{1/2},$$

which permits one to rewrite it in the form

$$\|v\|_{s,\xi} = [\xi]^s \left\| \varkappa_{[\xi]}^{-1} v \right\|_s. \tag{6.51}$$

By Lemma A.6 in the Appendix, the operators whose operator-valued symbols are uniformly bounded in such norms are bounded in $H^s(\mathbb{R}_+^{n+1})$.

For the case of general (with nonconstant coefficients and lower-order terms) elliptic equations

$$Du = f$$

on a manifold with boundary, one obtains the model operator by freezing coefficients at a point of the boundary and then by passing to the Fourier transform with respect to the boundary variables.

As a result, one arrives at a family of ordinary differential equations

$$\widehat{D}(x,\xi)\widetilde{u}(x,\xi,t) = \widetilde{f}(x,\xi,t), \quad t \in \mathbb{R}_+, \tag{6.52}$$

on the half-line with parameters $(x,\xi) \in T_0^*\partial M$. The cokernel of this family is always trivial (Eq. (6.52) is always solvable), and the kernel need not be trivial (but is always finite-dimensional, since the solution is uniquely determined by its jet of order $\operatorname{ord} D - 1$ at any given point t). Let us denote this kernel by $N(x,\xi)$. If there exists a smooth operator family

$$\widehat{B}(x,\xi) : H^s(\mathbb{R}_+) \longrightarrow E_x, \quad s \geq \operatorname{ord} D,$$

where E is some vector bundle over ∂M, such that the Schwartz kernel of $\widehat{B}(x,\xi)$ is smooth for $t > 0$ and

$$\widehat{B}(x,\xi) : N(x,\xi) \longrightarrow E_x$$

is an isomorphism, then the problem

$$\begin{cases} Du = f \in H^{s-m}(M) \\ \widehat{B}(x, -i\partial/\partial x)u = g \in H^{s-1/2-\operatorname{ord}\widehat{B}}(\partial M, E), \end{cases} \tag{6.53}$$

$m = \operatorname{ord} D$, for $u \in H^s(M)$ is Fredholm.

Remark 6.30. One can easily predict the Fredholm property of problem (6.53) on the basis of the fact that if we freeze the coefficients of D at the boundary in this problem, then the resulting problem can be obtained by the quantization $\xi \mapsto \partial/\partial x$ from the symbol

$$\begin{pmatrix} \widehat{D}(x,\xi) \\ \widehat{B}(x,\xi) \end{pmatrix},$$

which is invertible for $\xi \neq 0$. A rigorous proof includes the construction of a regularizer with the help of partitions of unity etc. (See the subsequent sections of this chapter.)

One usually takes the mapping $\widehat{B}$ in the form $\widehat{B} = i^*\widetilde{B}$, where i^* is the operator of restriction to the boundary $t = 0$ and $\widetilde{B}$ is a differential operator. Then (6.53) is a classical boundary value problem with the operator $\widetilde{B}$ in boundary conditions.

An example of an edge problem. Now let us consider an example of an edge problem very close to the preceding example 6.27. We essentially consider the same equation but in weighted Sobolev spaces rather than in ordinary Sobolev spaces.

In the half-space $\mathbb{R}^{n+1} \ni (x,r)$, consider the equation

$$Du \equiv \triangle u - \frac{a}{r^2}u = f,$$

where $\triangle$ is the Beltrami–Laplace operator and a is a nonnegative constant. In local coordinates, this equation has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial x^2} - \frac{a}{r^2}u = f.$$

Let us study the equation near the edge. To this end, we pass to the Fourier transform[5] with respect to the variables x. We obtain the family of equations

$$\widehat{D}(\xi)\widetilde{u} = \widetilde{f} \tag{6.54}$$

on the half-line $\mathbb{R}_+$, where

$$\widehat{D}(\xi) \equiv \sigma_\wedge(D)(\xi) = \frac{\partial^2}{\partial r^2} - |\xi|^2 - \frac{a}{r^2}$$

is the edge symbol of the operator D. In what spaces is it natural to study this family of equations?

We have agreed to seek the solution $u(x)$ of the equation itself in the edge Sobolev spaces $W^s(\mathbb{R}^{n+1}_+)$. After the Fourier transform from t to ξ, by the Parseval identity, the norm of the edge space acquires the form

$$\|u\|^2_{W^s(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^n_\xi \times \mathbb{R}^t_+} \left| \left(1 + |\xi|^2 + \frac{1}{t^2} - \frac{\partial^2}{\partial t^2}\right)^{s/2} \widetilde{u}(\xi, t) \right|^2 d\xi dt = \int_{\mathbb{R}^n_\xi} \|\widetilde{u}(\xi, \cdot)\|^2_{s,\xi}\, d\xi,$$

where the norm

$$\|\widetilde{v}\|^2_{s,\xi} = \int_0^\infty \left| \left(1 + |\xi|^2 + \frac{1}{t^2} - \frac{\partial^2}{\partial t^2}\right)^{s/2} \widetilde{v}(t) \right|^2 dt \tag{6.55}$$

is equivalent *for each given* ξ to the norm

$$\|v\|^2_s = \int_0^\infty \left| \left(1 - \frac{\partial^2}{\partial t^2} + \frac{1}{t^2}\right)^{s/2} v(t) \right|^2 dt.$$

The space with this norm will be denoted by $\mathcal{K}^s(\mathbb{R}_+)$. Thus it is natural to seek solutions of our family of equations (6.54) for each given ξ in the space $\mathcal{K}^s(\mathbb{R}_+)$ while watching that the integral over ξ expressing the norm of the original function u be convergent.

Let us find out whether this choice of spaces is natural from the viewpoint of the resulting model equation on the half-line itself.

For large r, the operator $\widehat{D}(\xi)$ is essentially an operator with constant coefficients, and it is natural to study it in the ordinary Sobolev spaces H^s. We can also rewrite the operator in the form

$$\widehat{D}(\xi) = -r^{-2}\left[\left(ir\frac{\partial}{\partial r}\right)^2 - i\left(ir\frac{\partial}{\partial r}\right) + |r\xi|^2 + a\right].$$

We see that for small r the operator is conically degenerate and can naturally be studied for these r in the weighted Sobolev spaces $H^{s,\gamma}$.

DEFINITION 6.31. The space $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ is glued from $H^{s,\gamma}(\mathbb{R}_+)$ for small r and $H^s(\mathbb{R}_+)$ for large r.

EXERCISE 6.32. (1) The spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{K}^{-s,-\gamma}(\mathbb{R}_+)$ are dual to each other with respect to the inner product in $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+, dr)$.

(2) One has $\mathcal{K}^s(\mathbb{R}_+) = \mathcal{K}^{s,s}(\mathbb{R}_+)$.

(3) The norm (6.55) possesses a homogeneity property similar to (6.50) and can be represented in a form similar to (6.51).

[5]The coefficients of this equation have already been frozen.

Thus the spaces $\mathcal{W}^s$, from which we have obtained the spaces $\mathcal{K}^s$ with the help of the Fourier transform, are indeed natural. However, we see that the model equation can be studied not only in the spaces $\mathcal{K}^s$ but also in the spaces $\mathcal{K}^{s,\gamma}$ with $s \neq \gamma$, which will give new edge spaces $\mathcal{W}^{s,\gamma}$ that include $\mathcal{W}^s$ as the special case with $s = \gamma$.

Thus we consider the operator family

$$\widehat{D}(\xi) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+).$$

We are interested in its kernel and cokernel, so we shall consider also the adjoint family

$$\widehat{D}(\xi)^* : \mathcal{K}^{2-s,2-\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+),$$

which is given on $C_0^\infty(\mathbb{R}_+)$ by the same differential expression as $\widehat{D}(\xi)$. We wish to solve the model example explicitly, and so in what follows we assume that $a = 0$.

To obtain a Fredholm problem for the original operator, we must ensure the invertibility of the operator $\widehat{D}(\xi)$ for large ξ.

Since the edge symbol is twisted-homogeneous, it suffices to do this for $|\xi| = 1$ and then extend the result to all $\xi \neq 0$ by homogeneity.

The conormal symbol

$$\sigma_c(\widehat{D}(\xi)) = -p^2 + ip$$

of the edge symbol in question is invertible for $p \neq 0, i$, so that the operator $\widehat{D}(\xi)$ is Fredholm for $\gamma \neq \frac{1}{2}, \frac{3}{2}$.

Let us compute the kernel and cokernel of the edge symbol for these γ.

Formally, the kernel of $\widehat{D}(\xi)$ and $\widehat{D}(\xi)^*$ is given by the expression

$$u = C_1 e^{-r} + C_2 e^{r}.$$

However, the constant C_2 is always zero, which follows from the integrability at infinity.

Next, the presence of the weight factor $r^{-2\gamma}$ in the definition of the norm in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ results in the assertion that $e^{-r} \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ if and only if $\gamma < \frac{1}{2}$.

Now let us exhaust all possible cases.

1. If $\gamma < \frac{1}{2}$, then $\widehat{D}(\xi)$ has the one-dimensional *kernel* $\left\{Ce^{-r}\right\}$ and a trivial cokernel. To make the edge symbol invertible, one can equip it, say, with a *condition* of the form

$$\int_0^\infty \phi(r)\widetilde{u}(r)\,dr = g,$$

where $\phi \in C_0^\infty(\mathbb{R}_+)$ is a given function nonorthogonal to e^{-r}.

2. For $\frac{1}{2} < \gamma < \frac{3}{2}$, the edge symbol $\widehat{D}(\xi)$ is invertible.

3. If $\gamma > \frac{3}{2}$, then the edge symbol $\widehat{D}(\xi)$ has a trivial kernel and the one-dimensional *cokernel* $\left\{Ce^{-r}\right\}$. To make the edge symbol invertible, we equip it with a *co-condition*, including a numerical unknown $\mu \in \mathbb{C}$:

$$\widehat{D}(\xi)\widetilde{u} + \mu\phi = \widetilde{f}.$$

Here ϕ is a given function satisfying the same nonorthogonality condition as above.

In all three cases, adding a finite-dimensional (in this example, one-dimensional) condition or co-condition where necessary, we have made the edge symbol for $|\xi| = 1$ an *invertible operator* $A(x, \xi)$. For $\gamma < 1/2$, this operator has the form

$$A(x,\xi) = \begin{pmatrix} \widehat{D}(\xi) \\ \widehat{B} \end{pmatrix} : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \begin{matrix} \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{6.56}$$

where

$$\widehat{B}v = \int_0^\infty \phi(r)v(r)\,dr.$$

For $\frac{1}{2} < \gamma < \frac{3}{2}$, we have

$$A(x,\xi) = \widehat{D}(\xi) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+). \tag{6.57}$$

Finally, for $\gamma > \frac{3}{2}$ the operator $A(x, \xi)$ is given by the formula

$$A(x,\xi) = \begin{pmatrix} \widehat{D}(\xi), & \widehat{C} \end{pmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix} \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+), \tag{6.58}$$

where

$$\widehat{C}\mu = \mu\phi(r).$$

Now we extend the operator-valued symbol $A(x, \xi)$ to all values $\xi \neq 0$ by twisted homogeneity. Here we assume that the group $\varkappa_\lambda$ acts trivially (as the identity operator) on the one-dimensional complex space $\mathbb{C}$. The component B (or C) of the symbol will be taken homogeneous of degree l, which does not necessarily coincide with m. (In fact, we deal with homogeneity in the sense of Douglis–Nirenberg.) By extending the symbol $\widehat{D}$ of the main operator to all $\xi \neq 0$, we obtain the same symbol, since it is twisted-homogeneous. The continuation of the symbols $\widehat{B}$ and $\widehat{C}$ gives the symbols

$$\begin{aligned} \widehat{B}(x,\xi)v &= |\xi|^{l+1/2} \int_0^\infty \phi(r|\xi|)v(r)\,dr, \\ \widehat{C}(x,\xi)\mu &= \mu|\xi|^{l+1/2}\phi(r|\xi|). \end{aligned}$$

By applying the inverse Fourier transform, we obtain the following problems for the operator D:

$$\begin{aligned} &Du = f, \quad \int_0^\infty \phi\left(r\left|-i\frac{\partial}{\partial x}\right|\right)\left|-i\frac{\partial}{\partial x}\right|^{l+1/2} u(x,r)\,dr = g(x), && \gamma < 1/2; \\ &Du = f, && \frac{1}{2} < \gamma < \frac{3}{2}; \\ &Du + \phi\left(r\left|-i\frac{\partial}{\partial x}\right|\right)\left|-i\frac{\partial}{\partial x}\right|^{l+1/2} \mu(x) = f, && \gamma > \frac{3}{2}. \end{aligned}$$

In what spaces will these problems be Fredholm? If we take the weight exponent γ in the model Fourier-problem to be equal to the smoothness exponent s (i.e., if we study the model problem in the space $\mathcal{K}^s$), then the Fredholm property holds provided that we assume that $u \in \mathcal{W}^s$, $f \in \mathcal{W}^{s-m}$, and the functions $g(x)$ and $\mu(x)$, if they occur in the problem, lie in an ordinary Sobolev space on the boundary.

EXERCISE 6.33. Compute the smoothness exponent of this Sobolev space.

However, the choice $\gamma = s$ is not always possible. (Namely, for $s = 1/2$ or $3/2$ the edge symbol cannot be made invertible in $\mathcal{K}^s$ by the addition of finitely many conditions.) In this case, we necessarily have to take $\gamma \neq s$, and the above-mentioned problems for our operator prove to be Fredholm in some more general spaces introduced in the next item.

The general case. We follow a scheme for studying elliptic equations on manifolds with multidimensional singularities which was apparently suggested for the first time in the case of Sobolev problems in (Sternin 1967).

We say that an operator P is *formally elliptic* if its principal symbol is invertible everywhere outside the zero section. We shall study the formally elliptic equation $Pu = f$ in the spaces $\mathcal{W}^s(\mathcal{M})$ (as well as in slightly more general spaces including a power-law weight in a neighborhood of the edge, see below). Our main task is to find out whether the equation is Fredholm and, if this is not the case, whether it is possible to impose some conditions on the solution u and the right-hand side f to make the equation Fredholm. A traditional way to prove the Fredholm property is to construct a regularizer. It is usually constructed locally, and then the local regularizers are glued together with the help of a partition of unity. Consider a formally elliptic edge-degenerate differential operator P of order m on a manifold $\mathcal{M}$ with edge X. In a neighborhood of any interior point, formal ellipticity coincides with the ordinary ellipticity, and one can construct the regularizer by usual methods of elliptic theory in the form of a pseudodifferential operator whose symbol is the inverse of the principal symbol $\sigma(P)$. Now consider a neighborhood of some point of the edge. Locally, we can assume that $X = \mathbb{R}^n$. Let us freeze the coefficients $a_{\alpha\beta l}$ of the operator P at some point of the edge and proceed in the equation $Pu = f$ to the Fourier transform with respect to the variables x (i.e., the variables along the edge). Then we obtain the family of equations

$$\sigma_\wedge(P)(x,\xi)\widetilde{u} = \widetilde{f}, \tag{6.59}$$

where the tilde stands for the Fourier transform with respect to the edge variables. To construct a regularizer of the operator P in a neighborhood of the edge, we should exactly solve Eq. (6.59) in the domain $r < \varepsilon$, where ε is a given (independent of r) positive number, for any $(x,\xi) \in T_0^*X$ (or at least for large $|\xi|$). The edge symbol possesses the easy-to-verify homogeneity property

$$\sigma_\wedge(P)(x,\lambda\xi) = \lambda^m \varkappa_\lambda \sigma_\wedge(P)(x,\xi)\varkappa_\lambda^{-1}, \tag{6.60}$$

and hence it suffices to solve (6.59), say, only for $|\xi| = 1$ on the *entire infinite cone* K_Ω. (Indeed, any solution of (6.59) has the form

$$\widetilde{u}(r,\omega) = |\xi|^{-m} v(|\xi| r, \omega), \tag{6.61}$$

where $v(r,\omega)$ is a solution of the equation

$$\sigma_\wedge(P)(x,\xi/|\xi|)v(r,\omega) = \widetilde{f}(r/|\xi|,\omega), \tag{6.62}$$

so that for arbitrarily large $|\xi|$, to know the solution of Eq. (6.59) in the domain $r \le \varepsilon$, one should know the solution of (6.62) for arbitrarily large r.)

Thus let us study Eq. (6.59) on the cone K_Ω. This is an elliptic equation that is cone-degenerate at the vertex of the cone K_Ω and behaves for large r as a usual elliptic equation with slowly varying coefficients at infinity in $\mathbb{R}^{k+1}$ written in polar coordinates.[6] What can one say about its solutions? Since the equation has variable coefficients, one cannot hope in general that it is uniquely solvable (invertible); we can only hope that it is Fredholm in suitable spaces, and then one can make it uniquely solvable by adding finitely many conditions. As we return from the model equation to the original equation with the operator P with the help of the inverse Fourier transform, these conditions become some conditions on the solution and the right-hand side of the original equation providing its Fredholm property. Thus first of all we should study the Fredholm property of the symbol $\sigma_\wedge(P)(x,\xi)$ on the infinite cone K_Ω in the spaces $\mathcal{K}^s(K_\Omega)$ obtained from $\mathcal{W}^s(\mathcal{M})$ by the Fourier transform with respect to the edge variables by analogy with the construction in the preceding item.

Let us write out the norm in $\mathcal{K}^s(\Omega)$. For functions supported in a neighborhood of the edge, we have

$$\begin{aligned}\|u\|^2_{\mathcal{W}^s} &= \int_W |(1+r^{-2}-\triangle_W)^{s/2}u|^2 d\mathrm{vol}\\ &= \int_{\mathbb{R}^n}\int_{K_\Omega} |(1+r^{-2}-\triangle_x-\triangle_{K_\Omega})^{s/2}u|^2 r^k\,dr\,d\omega\,dx\\ &= \int_{\mathbb{R}^n}\int_{K_\Omega} |(1+r^{-2}+|\xi|^2-\triangle_{K_\Omega})^{s/2}\widetilde{u}|^2 r^k\,dr\,d\omega\,d\xi.\end{aligned} \tag{6.63}$$

It is the integral over K_Ω that defines (modulo an equivalence depending on ξ) the squared norm in $\mathcal{K}^s(K_\Omega)$:

$$\|v\|_{\mathcal{K}^s(K_\Omega)} = \left\{\int_{K_\Omega} |(1+r^{-2}-\triangle_{K_\Omega})^{s/2}v|^2 r^k\,dr\,d\omega\right\}^{1/2}.$$

Making the change of variable $r\mapsto[\xi]\,r$ in the integral in the last line in (6.63), we obtain the following assertion.

PROPOSITION 6.34. *The norm in $\mathcal{W}^s(W)$ can be represented in the form*

$$\|u\|_s = \left(\int [\xi]^{2s}\left\|\varkappa^{-1}_{[\xi]}\widetilde{u}\right\|^2_{\mathcal{K}^s(K_\Omega)} d\xi\right)^{1/2}. \tag{6.64}$$

We study the Fredholm property of the edge symbol by constructing a regularizer. Starting from the structure of the edge symbol, let us find out in what spaces one can construct a regularizer and whether the spaces $\mathcal{K}^s(K_\Omega)$ are suitable for this. We cover the cone K_Ω by two overlapping neighborhoods, a neighborhood U_1 of the vertex $r=0$ and a neighborhood U_2 of infinity. One can construct the regularizer locally, in each of these neighborhoods. In U_2, our operator is a usual elliptic operator with coefficients slowly varying at infinity, and hence the regularizer can be constructed in the usual Sobolev spaces $H^s(U_2)$. In U_1, our operator is cone-degenerate, and hence the regularizer, in accordance with the material of the preceding chapter, can be constructed in weighted Sobolev spaces $H^{s,\gamma}(U_1)$. On the intersection $U_1\cap U_2$ these spaces coincide,

$$H^{s,\gamma}(U_1\cap U_2) = H^s(U_1\cap U_2), \tag{6.65}$$

[6]An operator with coefficients slowly varying at infinity in $\mathbb{R}^{k+1}$ is an operator of the form

$$D = \sum a_\alpha(y)(-i\partial/\partial y)^\alpha,$$

where the coefficients $a_\alpha(y)$, $y\in\mathbb{R}^{k+1}$, satisfy the estimates

$$|a^{(\beta)}_\alpha(y)| \le \mathrm{const}(1+|y|)^{-|\beta|}.$$

Regularizers for such operators will be discussed later on in this chapter.

and so we can glue them together by using a partition of unity subordinate to the cover (U_1, U_2). The resulting space will be denoted by $\mathcal{K}^{s,\gamma}(K_\Omega)$.

EXERCISE 6.35. Show that $\mathcal{K}^s(K_\Omega) = \mathcal{K}^{s,s}(K_\Omega)$, just as in the example considered above.

For the norm in $\mathcal{K}^{s,\gamma}(K_\Omega)$ one can readily write out a simple equivalent global expression that does not use partitions of unity. Namely, consider the space $L^2(K_\Omega, r^k\, dr\, d\omega)$ on the cone K_Ω, where

$$r^k\, dr\, d\omega, \quad k = \dim \Omega \tag{6.66}$$

is the Riemannian volume form corresponding to the metric

$$dr^2 + r^2 d\omega^2 \tag{6.67}$$

on K_Ω. Consider also a weight function $\rho(r)$ equal to r in a neighborhood of $r = 0$ and tending to 1 as $r \to \infty$. The following assertion holds.

PROPOSITION 6.36. *An equivalent norm in $\mathcal{K}^{s,\gamma}(K_\Omega)$ can be given by the formula*

$$\|u\|_{s,\gamma} = \left\| (1 + r^{-2} + \triangle_{K_\Omega})^{s/2} \rho^{\gamma/2} u \right\|_{L^2(K_\Omega)}, \tag{6.68}$$

where $\triangle_{K_\Omega}$ is the Beltrami–Laplace operator on K_Ω associated with the cone-degenerate metric (6.67).

Thus we shall study the edge symbol in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$. The following theorem holds.

THEOREM 6.37. (1) *The edge symbol of an operator P of order m is continuous in the spaces*

$$\sigma_\wedge(P)(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega). \tag{6.69}$$

(2) *If the operator P is formally elliptic, then its edge symbol is Fredholm in the spaces* (6.69) *if and only if the conormal symbol*[7]

$$\sigma_c(\sigma_\wedge(P))(p) : H^s(\Omega) \longrightarrow H^{s-m}(\Omega) \tag{6.70}$$

of the edge symbol is invertible on the weight line $\operatorname{Im} p = \gamma$. Moreover, the kernel and cokernel of the operator (6.69) *are independent of the smoothness exponent s.*

The first assertion of the theorem is obvious, and the second assertion can be proved by the construction of a regularizer in accordance with the scheme given above.

Boundary and coboundary conditions on the edge. Now assume that we have managed to find a weight exponent γ in such a way that the edge symbol is Fredholm for all $(x,\xi) \in T_0^*X$. For each (x,ξ) we can make it an invertible operator by subjecting the solution and the right-hand sides of Eq. (6.59) to finitely many orthogonality conditions

$$(\widetilde{u}, b_j) = 0, \quad j = 1, ..., M; \qquad (\widetilde{f}, c_j^*) = 0, \quad j = 1, ..., N. \tag{6.71}$$

[7] One readily sees that it is independent of ξ.

(the inner product is taken in the space $L^2(K_\Omega)$), and the functions b_j and c_j^* can be chosen to be infinitely smooth:

$$b_j \in \bigcap_s \mathcal{K}^{s,-\gamma}(K_\Omega), \quad c_j^* \in \bigcap_s \mathcal{K}^{s,m-\gamma}(K_\Omega). \tag{6.72}$$

We can pass from problem (6.59), (6.71) to the nonhomogeneous problem

$$\sigma_\wedge(P)(x,\xi)\widetilde{u} + C\widetilde{v} = \widetilde{f}, \tag{6.73}$$

$$B\widetilde{u} = \widetilde{g}, \tag{6.74}$$

where the operators

$$B : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathbb{C}^M, \qquad C : \mathbb{C}^N \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \tag{6.75}$$

are given by the formulas

$$B\widetilde{u} = \big((\widetilde{u}, b_1), \dots, (\widetilde{u}, b_M)\big), \quad C\widetilde{v} = \widetilde{v}_1 c_1 + \dots + \widetilde{v}_N c_N \tag{6.76}$$

in which

$$c_1, \dots, c_N \in \bigcap_s \mathcal{K}^{s,\gamma-m}(K_\Omega) \tag{6.77}$$

are functions such that

$$\det \begin{pmatrix} (c_1, c_1^*) & \dots & (c_1, c_N^*) \\ \dots & \dots & \dots \\ (c_N, c_1^*) & \dots & (c_N, c_N^*) \end{pmatrix} \neq 0. \tag{6.78}$$

Now suppose that these conditions and co-conditions can be chosen in such a way that they continuously depend on (x,ξ) and, together with the edge symbol, form a family of invertible operators

$$\begin{pmatrix} \sigma_\wedge(P)(x,\xi) & C(x,\xi) \\ B(x,\xi) & 0 \end{pmatrix} : \mathcal{K}^{s,\gamma}(K_\Omega) \oplus E_x \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \oplus F_x, \tag{6.79}$$

where E and F are finite-dimensional vector bundles over X. (This is not always possible; the obstruction is computed in the subsequent chapters.) Since the edge symbol is twisted homogeneous, we can assume without loss of generality that the operators (6.75) are also twisted-homogeneous in ξ (the group $\varkappa_\lambda$ acts in the fibers of E and F as the identity operator by definition):

$$B(x,\lambda\xi) = \lambda^l B(x,\xi)\varkappa_\lambda^{-1}, \quad C(x,\lambda\xi) = \lambda^{l'} \varkappa_\lambda C(x,\xi). \tag{6.80}$$

(It suffices to define these operators for $|\xi| = 1$ and then extend them by continuity.)

Passing to the inverse Fourier transform and unfreezing the coefficients, from the family (6.79) we obtain the *edge problem*

$$Pu + C(x, -i\partial/\partial x)v = f, \tag{6.81}$$

$$B(x, -i\partial/\partial x)u = g, \tag{6.82}$$

where v and g are sections of the bundles E and F on X.

Here $\widehat{B} = B(x, -i\partial/\partial x)$ is an operator of *edge boundary conditions* (or a *boundary operator* for short[8]), which takes a function $u(x, r, \omega)$ on $\mathcal{M}$ to the function

$$[\widehat{B}u](x) = \int e^{ix\xi} \int_{K_\omega} \vec{b}(x, \xi, r\xi, \omega)\widetilde{u}(\xi, r, \omega) r^{k+1} dr d\omega d\xi \tag{6.83}$$

on the boundary, and $\widehat{C} = C(x, -i\partial/\partial x)$ is an operator of *edge coboundary conditions*, or a *coboundary operator*. (The adjoint of $\widehat{C}$ has the form (6.83).)

EXERCISE 6.38. Write out the operator $\widehat{C}$ explicitly.

The finiteness theorem. In what spaces is problem (6.81), (6.82) Fredholm? If the weight exponent γ is equal to the smoothness exponent s, then, by the preceding (Eq. (6.64)) these are just the spaces $\mathcal{W}^s(\mathcal{M})$. In other words, if we can take $\gamma = s$, the problem (6.81), (6.82) can be considered in the spaces $\mathcal{W}^s(\mathcal{M})$. However, one cannot always take $s = \gamma$, and hence we introduce spaces $\mathcal{W}^{s,\gamma}(\mathcal{M})$ with $s \neq \gamma$ by letting the weight exponent in the formula (6.64) take arbitrary values.

DEFINITION 6.39. By $\mathcal{W}^{s,\gamma}(W)$ we denote the space of functions on W with the norm

$$\|u\|_{s,\gamma} = \left(\int [\xi]^{2s} \left\| \varkappa_{[\xi]}^{-1} \widetilde{u} \right\|^2_{\mathcal{K}^{s,\gamma}(K_\Omega)} d\xi \right)^{1/2}. \tag{6.84}$$

The space $\mathcal{W}^{s,\gamma}(\mathcal{M})$ is obtained from $\mathcal{W}^{s,\gamma}(W)$ by the standard gluing with the Sobolev space $H^s(M)$ in the interior of M.

We shall consider problem (6.81), (6.82) in the spaces

$$\begin{pmatrix} P & C(x, -i\partial/\partial x) \\ B(x, -i\partial/\partial x) & 0 \end{pmatrix} : \mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^{s-m+l'+(k+1)/2}(X, E) \\ \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \oplus H^{s-l-(k+1)/2}(X, F). \tag{6.85}$$

PROPOSITION 6.40. *The operator* (6.85) *is continuous.*

Proof. This follows from Lemma A.6, since the estimates on symbols required in the assumptions of this lemma are a consequence of the twisted homogeneity. □

Now we are in a position to state the finiteness theorem.

THEOREM 6.41. *Suppose that the operator P is formally elliptic and the edge symbol of problem* (6.81), (6.82) *is invertible in the spaces*

$$\begin{pmatrix} \sigma_\wedge(P)(x,\xi) & C(x,\xi) \\ B(x,\xi) & 0 \end{pmatrix} : \mathcal{K}^{s,\gamma}(K_\Omega) \oplus E_x \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \oplus F_x.$$

Then this problem is Fredholm in the spaces (6.85).

[8] We use terminology introduced for the case of Sobolev problems in (Sternin 1967).

The proof of the finiteness theorem is based on the calculus of edge pseudodifferential operators developed further in this chapter.

Here we only note that the proof involves the construction of a regularizer. Let us state an assertion concerning the properties of this regularizer, restricting ourselves for simplicity to the case in which there are neither boundary nor coboundary conditions.

PROPOSITION 6.42. *Suppose that the edge symbol of a formally elliptic differential operator P of order m is invertible in the spaces*

$$\sigma_\wedge(P)(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

for all $\xi \neq 0$. Then there exists a number $\varepsilon > 0$ and an operator (*right regularizer*)

$$R : \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \longrightarrow \mathcal{W}^{s,\gamma}(\mathcal{M})$$

such that

$$RD = 1 + Q,$$

where the remainder Q is continuous in the spaces

$$Q : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s+\varepsilon,\gamma+\varepsilon}(\mathcal{M})$$

(*and, in particular, is compact in $\mathcal{W}^{s,\gamma}(\mathcal{M})$*). *A similar assertion is valid for the left regularizer.*

COROLLARY 6.43. *The kernel and cokernel of the problem are independent of the smoothness exponent s.*

The proof will be given later in this chapter. So far, we only note that the number ε is determined by the width of the strip of values of γ in which the edge symbol remains invertible.

Appendix A

Fiber Bundles and Direct Integrals

Here we give a simple general scheme[1] that permits one to describe Hilbert function spaces on the total spaces of locally trivial bundles

$$\pi : Y \longrightarrow X$$

with compact base as spaces of Hilbert-valued functions (or distributions) on X. Moreover, "along X" our spaces are similar to usual Sobolev spaces. (There are no distinguished or degenerate directions.) It is convenient to define such spaces in the Fourier transform as direct integrals of Hilbert spaces. We also establish a boundedness theorem for pseudodifferential operators in such spaces.

Throughout the following, we use the notation $[\xi] = (1 + |\xi|^2)^{1/2}$.

A.1. Local theory

Let H be a Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot\,,\cdot)$. Suppose that it is equipped with a family $\{\| \cdot \|_\xi\}$ of Hilbert norms depending on the parameter $\xi \in \mathbb{R}^n$, equivalent to the original norm $\| \cdot \|$, and satisfying the following conditions:

- $\|u\|_\xi$ is a measurable function of ξ for any $u \in H$;
- there exist constants C and N such that

$$\|u\|_\eta \le C \left(\frac{[\eta]}{[\xi]} + \frac{[\xi]}{[\eta]} \right)^N \|u\|_\xi \quad \text{for any } \xi, \eta \in \mathbb{R}^n \text{ and } u \in H,\ u \neq 0. \tag{A.1}$$

EXERCISE A.1. (1) Condition (A.1) is invariant with respect to affine changes of the variables ξ. (The number N is preserved, and the constant C may vary.)

(2) One has the inequality

$$\|u\|_{\xi+\eta} \le C_1 \, [\eta]^N \, \|u\|_\xi .$$

◀ (1) The invariance with respect to linear transformations is obvious. As to translations and assertion (2), the desired results follow from the Peetre inequalities

$$\frac{[\xi]}{[\lambda]} \le [\xi \pm \lambda] \le [\xi]\,[\lambda]\,.$$

▶

[1] In the final version of the book, we intend to include this Appendix in Chapter 4, which deals with pseudodifferential operators.

DEFINITION A.2. A family of norms satisfying the above-mentioned conditions will be called a *tempered family of norms*.

We denote the space H equipped with the norm $|\!\!| \cdot |\!\!|_\xi$ by H_ξ and define the *direct integral* of H_ξ by the formula

$$\int H_\xi \, d\xi \stackrel{\text{def}}{=} \left\{ u : \mathbb{R}^n \longrightarrow H \mid u(\xi) \text{ is measurable and } \int |\!\!| u(\xi) |\!\!|_\xi^2 \, d\xi < \infty \right\}. \tag{A.2}$$

Remark A.3. (1) Strong and weak measurability are the same in a separable Hilbert space, so we do not specify the kind of measurability in the definition.

(2) If $u(\xi)$ is measurable, then so is $|\!\!| u(\xi) |\!\!|_\xi$, and hence (A.2) is well defined.

The space $\mathcal{H} = \int H_\xi \, d\xi$ is equipped with the natural norm

$$\|u\| = \left\{ \int |\!\!| u(\xi) |\!\!|_\xi^2 \, d\xi \right\}^{1/2}, \tag{A.3}$$

which makes it a Hilbert space.

LEMMA A.4. *The space $\mathcal{H}$ is the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^n, H)$ with respect to the norm* (A.3).

The proof is standard.

We are actually interested in the space $\mathcal{F}^{-1}\mathcal{H}$ obtained from $\mathcal{H}$ by the inverse Fourier transform. It follows from the preceding lemma that $\mathcal{F}^{-1}\mathcal{H}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n, H)$ with respect to the norm

$$\|u\|_{(0)} = \|\widetilde{u}\| .$$

EXAMPLE A.5. If $H = \mathbb{C}$ and $|\!\!| u |\!\!|_\xi = [\xi] |u|$, then $\mathcal{F}^{-1}\mathcal{H} = H^s(\mathbb{R}^n)$ is the ordinary Sobolev space in $\mathbb{R}^n$.

If $H = H^s(M)$, where M is a smooth compact manifold, and

$$|\!\!| u |\!\!| = \left\| \left(\frac{1 + |\xi|^2 - \triangle}{1 - \triangle} \right)^{s/2} u \right\|_{H^s(M)},$$

where $\triangle$ is the Beltrami–Laplace operator on M, then $\mathcal{F}^{-1}\mathcal{H} = H^s(\mathbb{R}^n \times M)$ is the Sobolev space on $\mathbb{R}^n \times M$.

Let

$$\mathcal{G} = \int G_\xi \, d\xi$$

be another direct integral of Hilbert spaces.

Let us study the continuity of pseudodifferential operators from $\mathcal{F}^{-1}\mathcal{H}$ to $\mathcal{F}^{-1}\mathcal{G}$. We consider operator-valued symbols $D(x, \xi)$ ranging in $\mathcal{L}(H, G)$ and satisfying the estimates

$$\left\| \frac{\partial^\alpha D}{\partial x^\alpha}(x, \xi) : H_\xi \longrightarrow G_\xi \right\| \leq C_{\alpha l} \, [x]^{-l}, \quad l, |\alpha| = 0, 1, 2, \dots . \tag{A.4}$$

LEMMA A.6. *If the estimates* (A.4) *hold, then the operator*

$$D\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right) : \mathcal{F}^{-1}\mathcal{H} \longrightarrow \mathcal{F}^{-1}\mathcal{G}$$

is continuous.

Proof. We shall prove an equivalent assertion, namely, the continuity of the operator

$$\widehat{D} = D\left(i\overset{2}{\frac{\partial}{\partial \xi}}, \overset{1}{\xi}\right) : \mathcal{H} \longrightarrow \mathcal{G}.$$

The operator acts by the formula

$$\left[\widehat{D}u\right](\xi) = \int \widetilde{D}(\xi - \eta, \eta) u(\eta)\, d\eta = \int \widetilde{D}(z, \xi - z) u(\xi - z)\, dz, \tag{A.5}$$

where $\widetilde{D}$ is the Fourier transform of the symbol D with respect to the first argument. By virtue of the estimates (A.4), the Fourier transform is continuous and satisfies the estimates

$$\left\| \widetilde{D}(z, \eta) : H_\eta \longrightarrow G_\eta \right\| \leq C_l \, [z]^{-l}, \quad l = 0, 1, 2, \dots .$$

We can rewrite (A.5) in the form

$$\widehat{D}u = \int U(\,\cdot\,, z)\, dz, \tag{A.6}$$

where

$$U(\xi, z) = \widetilde{D}(z, \xi - z) u(\xi - z)$$

By virtue of the properties of the norm and the estimates imposed on $\widetilde{D}$, we have

$$|U(\xi, z)|_\xi \leq \text{const}\, [z]^N \, |U(\xi, z)|_{\xi - z} \leq \text{const}\, [z]^{-M} \, |u(\xi - z)|_{\xi - z},$$

where M is arbitrarily large. Hence

$$\begin{aligned} \|U(\,\cdot\,, z)\| &= \left\{ \int |U(z, \xi|^2 d\xi \right\}^{1/2} \\ &\leq \text{const}\, [z]^{-M/2} \left\{ \int |u(\xi - z)|^2_{\xi - z} dz \right\}^{1/2} = \text{const}\, [z]^{-M/2} \, \|u\| \, ; \end{aligned}$$

by substituting this estimate into (A.6) and by integrating over z, we arrive at the desired result. □

COROLLARY A.7. *The operator of multiplication by a smooth compactly supported function is continuous in the space* $\mathcal{F}^{-1}\mathcal{H}$.

A.2. Globalization

So far we have defined spaces of Hilbert-valued functions on $\mathbb{R}^n$. To proceed to a manifold, we should study how the norms are affected by changes of variables. Let $|\cdot|$ be a given tempered family of norms in a Hilbert space H, and let $\|\cdot\|$ be the corresponding norm in the direct integral $\mathcal{H}$ of the spaces H_ξ.

PROPOSITION A.8. *Let $U \subset \mathbb{R}^n$ be a bounded domain, and let $f : U \longrightarrow V \subset \mathbb{R}^n$ be a diffeomorphism extendible into some neighborhood of the closure $\overline{U}$ of the domain U. Then there exist positive constants C_1 and C_2 such that the inequalities*

$$c\,\|\widetilde{u}\| \leq \left\|\widetilde{f^*u}\right\| \leq \|\widetilde{u}\|$$

(*where the tilde stands for the Fourier transform*) *hold for each smooth H-valued function $u(x)$ supported in V.*

Proof. Let u be an H-valued function supported in V, let $v(\xi)$ be its Fourier transform, and let $w(\eta)$ be the Fourier transform of f^*u. Next, let $\rho(y)$ be a smooth compactly supported function whose support is contained in the domain of the extended diffeomorphism f and which is equal to unity in U. Then

$$w(\eta) = \left(\frac{1}{2\pi}\right)^n \iint e^{i(f(y)\xi - y\eta)}\rho(y)v(\xi)\,d\xi dy. \tag{A.7}$$

Using the operator

$$L = \frac{1 - i\langle f'(y)\xi - \eta, \partial/\partial y\rangle}{1 + |f'(y)\xi - \eta|^2},$$

which satisfies the relation

$$Le^{i(f(y)\xi - y\eta)} = e^{i(f(y)\xi - y\eta)},$$

and integrating by parts M times in (A.7), we obtain

$$w(\eta) = \left(\frac{1}{2\pi}\right)^n \iint e^{i(f(y)\xi - y\eta)}[({}^tL)^M\rho(y)]v(\xi)\,d\xi dy, \tag{A.8}$$

where ${}^tL$ is the transpose of L. Moreover, the estimate

$$|({}^tL)^M\rho(y)| \leq \text{const}\big[1 + |f'(y)\xi - \eta|^2\big]^{-M/2}$$

is obviously valid. In the integral (A.8), we make the change of variables

$$\xi = \xi(\eta, t) \equiv f'(y)^{-1}(t + \eta);$$

then it becomes

$$w(\eta) = \left(\frac{1}{2\pi}\right)^n \iint e^{i(f(y)\xi(\eta,t) - y\eta)}(\det f'(y))^{-1}[({}^tL)^M\rho(y)]v(f'(y)^{-1}(t+\eta))\,dtdy, \tag{A.9}$$

or

$$w(\eta) = \int U(\eta, t, y)\,dtdy,$$

where the integrand U in (A.9) satisfies the estimate

$$|U(\eta, t, y)|_\eta \leq \text{const}\,[t]^{(N-M)/2}\,|v(f'(y)^{-1}(t+\eta))|_{f'(y)^{-1}(t+\eta)}.$$

Now the end of the proof is the same as in Lemma A.6 from the preceding subsection. □

This proposition, in conjunction with the corollary to Lemma A.6 in the preceding section, shows that by using partitions of unity we can introduce a well-defined space $\mathcal{H}(X) \equiv \mathcal{H}(X, \| \cdot \|)$ of H-valued functions on X whose local model is the space $\mathcal{H}$ constructed in the preceding section. Pseudodifferential operators whose complete symbols in local coordinates satisfy the estimates (A.4) are continuous in such spaces.

In this connection, we slightly modify the definition of symbols with compact fiber variation given in Chap. 4.

DEFINITION A.9. Let $\| \cdot \|_\xi^1$ and $\| \cdot \|_\xi^2$ be two tempered families of norms in Hilbert spaces H_1 and H_2, respectively. By $S^0_{CV} \equiv S^0_{CV}(\mathbb{R}^{2n}, H_1, H_2)$ we denote the space of operator-valued symbols $a(x,\xi)$ ranging in $\mathcal{L}(H_1, H_2)$, satisfying the estimates

$$\left\| \frac{\partial^{\alpha+\beta} a(x,\xi)}{\partial x^\alpha \partial \xi^\beta} : H_{1\xi} \longrightarrow H_{2\xi} \right\| \le C_{\alpha\beta} [\xi]^{-|\beta|}, \tag{A.10}$$

and having compact fiber variation. The space $S^0_{CV}(T^*X, H_1, H_2)$ is defined as the space of operator-valued symbols whose coordinate representatives satisfy the estimates (A.10).

A.3. Versions of the Definition of the Norm

A tempered family of norms can always be given by the formula

$$\|u\|_\xi = \|A(\xi)u\|, \tag{A.11}$$

where $A(\xi)$ is a strongly measurable family of bounded operators in H and $\| \cdot \|$ is the norm in H.

EXERCISE A.10. Prove this.

The operator $A(\xi)$ is uniquely determined if we require that it be positive and self-adjoint. However, these requirements are not necessary in applications; moreover, one can proceed to equivalent norms.

LEMMA A.11. *Two tempered families of norms determined, in accordance with* (A.11)*, by operator families $A(\xi)$ and $B(\xi)$ determine the same (up to norm equivalence) space $\mathcal{H}$ if and only if the operator families*

$$A(\xi)B^{-1}(\xi), \quad B(\xi)A^{-1}(\xi)$$

are bounded uniformly with respect to ξ.

The proof is obvious.

Although the assertion is trivial, it permits one to give quite remarkable equivalent expressions for the norm. For example, let $\varkappa_\lambda$ be the group of dilations in $H^s(\mathbb{R}^k)$ acting by the formula

$$\varkappa_\lambda u(x) = \lambda^{k/2} u(\lambda x), \quad \lambda \in \mathbb{R}_+.$$

The operator families

$$A(\xi) = \left(\frac{1 + |\xi|^2 - \triangle}{1 - \triangle} \right)^{s/2}, \quad B(\xi) = [\xi]^s \varkappa^{-1}_{[\xi]}$$

satisfy the assumptions of the lemma, and hence the norms associated with these families are equivalent. Moreover, the family $B(\xi)$ defines the space $\mathcal{W}^s(\mathbb{R}^n, H^s(\mathbb{R}^k))$. We obtain the well-known identity

$$H^s(\mathbb{R}^{n+k}) = \mathcal{W}^s(\mathbb{R}^n, H^s(\mathbb{R}^k)).$$

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