Explicit formulation of the solution of Hamada-Leray-Wagschal's theorem

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ABSTRACT. – In this paper, an explicit formula of the solution of Hamada-Leray-Wagschal's theorem is given. For this, only structure's theorem of finite dimensional determination's function and linear algebra technics developped in [1] are used.

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1 Introduction

In a previous paper, the monodromy of the ramified Cauchy problem was studied [1]. The present paper is a direct application of previous methods in order to give an explicit formulation of the solution of Hamada-Leray-Wagschal's theorem [3, 4]. First, the problem is recalled. Let $a(x, D)$ a linear differential operator of order m

$$a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}^{n+1}$, that is $a_\alpha \in \mathbf{C}\{x\}$ (where $x = (x_0, \ldots, x_n)$ is the variable of $\mathbf{C}^{n+1}$ and $x' = (x_1, \ldots, x_n)$). Its principal symbol will be written $g(x, \xi)$ and we assume the hyperplane $S : x_0 = 0$ to be non-characteristic in the origin : this means that $g(0; 1, 0, \ldots, 0) \neq 0$. We will denote by Ω_0 an open neighbourhood of the origin of $\mathbf{C}^{n+1}$ such that all functions a_α are defined and holomorphic on Ω_0. We suppose that the operator $a(x, D)$ is

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an operator with multiple characteristics of constant multiplicity : this means that there are some functions $(x, \xi') \mapsto \lambda_i(x, \xi')$, where $i = 1, \ldots, d$, holomorphic in a neighbourhood of $(\overline{x} = 0,\ \overline{\xi}' = (1, 0, \ldots, 0))$ and some integers $m_i \geq 1$ such that

$$g(x, \xi) = \prod_{i=1}^{d} (\xi_0 - \lambda_i(x, \xi'))^{m_i} \text{ for } (x, \xi') \text{ closed to } (\overline{x}, \overline{\xi}')$$

and, if $\lambda_i = \lambda_i(\overline{x}, \overline{\xi}')$, $\lambda_i \neq \lambda_j$ if $i \neq j$.

Then, we can solve the non-linear first order Cauchy problem

$$\begin{cases} D_0 k_i(x) & = & \lambda_i(x, D' k_i(x)), \\ k_i(x) & = & x_1 \quad \text{for } x_0 = 0 \end{cases}$$

where $D' k_i(x) = (D_1 k_i(x), \ldots, D_n k_i(x))$. There is an unique solution of this problem in a neighbourhood of the origin. We have $D k_i(0) = (\lambda_i, 1, 0, \ldots, 0)$; thereby the functions k_i can be assumed defined and holomorphic on Ω_0 and $D k_i(x) \neq 0$ for $x \in \Omega_0$. This allows us to define hypersurfaces $K_i = \{x \in \Omega_0; k_i(x) = 0\}$; if T is the hyperplane $x_0 = x_1 = 0$ in S, then we have $K_i \cap S = \Omega_0 \cap T$: this means that the K_i are the characteristic hypersurfaces which go out from T.

Now, the Hamada-Leray-Wagschal's theorem is recalled. Consider the following problem :

$$\begin{cases} a(x, D) u(x) & = & \displaystyle\sum_{i=1}^{d} v_i(t, x)_{\big| t = k_i(x)}, \\ D_0^h u(x) & = & w_h(t, x)_{\big| t = x_1} \qquad \text{for } x_0 = 0,\ 0 \leq h < m. \end{cases} \tag{1}$$

In the sequel $\mathcal{R}(\dot{D}_\omega)$ will denote the universal covering of the open set $\dot{D}_\omega = \{t \in \mathbf{C};\ 0 < |t| < \omega\}$. Then, we have [3, 4] the

Theorem 1.1 *Let $\Omega \subset \Omega_0$ be a connected open neighbourhood of the origin of $\mathbf{C}^{n+1}$. So there is a connected open neighbourhood $\Omega' \subset \Omega$ of the origin of $\mathbf{C}^{n+1}$ and $\omega_0 > 0$ such that: let $0 < \omega \leq \omega_0$, $a \in S \cap \Omega'$ such that $0 < |a_1| < \omega$, v_i and w_h some holomorphic germs at $(t, x) = (a_1, a)$ which have a holomorphic extension on $\mathcal{R}(\dot{D}_\omega) \times \Omega$. Then the solution of (1) could be written*

$$u(x) = \sum_{i=1}^{d} u_i(t, x)_{\big| t = k_i(x)} \qquad \textit{for } x \textit{ close to } a$$

where u_i is a holomorphic germ at (a_1, a) which have a holomorphic extension on $\mathcal{R}(\dot{D}_\omega) \times \Omega'$.

The aim of this paper is to explicit the u_i's structure in function of v_i and w_h. More precisely, if the germs v_i and w_h are finite dimensional determination's functions, then u_i are finite dimensional determination's function too [1]. Using a structure's theorem of finite dimensional determination's functions in a punctured disk and technics of [1], the u_i's form will be precise.

2 Notations, recall and main theorem

Let X a connected complex manifold and E a complex Banach space. The complex vector space of holomorphic functions $f : X \to E$ will be written $\mathcal{H}(X;E)$ and $\mathcal{O}_a = \mathcal{O}_a(X;E)$ will be the complex vector space of holomorphic germs at point $a \in X$ with values in E.

If $u \in \mathcal{O}_a$ has an analytic continuation along a path $\gamma : I \to X$, where $I = [0,1]$, of origin a and endpoint b, we will write $u_\gamma \in \mathcal{O}_b$ the germ in b obtained by analytic continuation of u along γ.

$\Gamma_a = \Gamma_a(X)$ will be the space of loops $\gamma : I \to X$ of origin a and let $u \in \mathcal{O}_a$ be a germ which has an analytic continuation along all loops $\gamma \in \Gamma_a$. We denote by F_a^u the subvector space of $\mathcal{O}_a$ spanned by $(u_\gamma)_{\gamma\in\Gamma_a}$ of all determinations of u at point a and $A_\gamma^u \in GL(F_a^u)$ the automorphism

$$A_\gamma^u : \theta \in F_a^u \longmapsto \theta_\gamma \in F_a^u.$$

Since this automorphism depends only on the homotopy class $[\gamma]$ of the path $\gamma \in \Gamma_a$, we have a linear representation of Poincaré's group $\pi_1(X,a)$

$$A^u : [\gamma] \in \pi_1(X,a) \longmapsto A_\gamma^u \in GL(F_a^u) \text{ where } \gamma \in [\gamma];$$

this linear representation is called the monodromy of germ u.

The germ u is of finite dimensional determination if F_a^u is a finite dimensional vector space. The space $\mathcal{O}_a^f$ of germs of finite dimensional determination is a subvector space of $\mathcal{O}_a$. For $u \in \mathcal{O}_a^f$, $\sigma_\gamma(u)$ is the A_γ^u automorphism's spectrum. When u has a holomorphic extension on X, we have $u_\gamma = u$ for all $\gamma \in \Gamma_a$, so $F_a^u = \mathbf{C}u$ and A_γ^u is the identity map. When u is not the null germ, $\dim F_a^u = 1$ and $\sigma_\gamma(u) = \{1\}$; when $u = 0$, we will say that $\sigma_\gamma(u) = \emptyset$. At last, a subvector space F of $\mathcal{O}_a$ is said invariant under analytic continuation if, for all $u \in F$ and $\gamma \in \Gamma_a$, u has an analytic continuation along γ and $u_\gamma \in F$.

We recall now the following structure's theorem. We denote by δ the loop defined by $s \mapsto (a_1 e^{2i\pi s}, a)$ where $(a_1, a) \in \dot{D}_\omega \times \Omega$. Notice that δ is a generator loop of $\pi_1(\dot{D}_\omega \times \Omega, (a_1, a)) \simeq \mathbf{Z}$.

Theorem 2.1 *A germ* $u \in \mathcal{O}_{(a_1,a)}(\mathcal{R}(\dot{D}_\omega) \times \Omega)$ *is of finite dimensional determination if, and only if, it could be written*

$$u(t,x) = \sum_{m \in M} t^{p_m} \sum_{k=0}^{q_m} a_{mk}(t,x)[\ln t]^k \qquad (2)$$

where M *is a finite set,* $\mathcal{R}e(p_m) \in [0,1[$, $p_m \neq p_{m'}$ *if* $m \neq m'$ *and the functions* $a_{mk} : \dot{D}_\omega \times \Omega \to \mathbf{C}$ *are holomorphic,* $a_{mq_m} \not\equiv 0$.

Moreover, $\{\lambda_m = e^{2i\pi p_m}\}_{m \in M}$ *are the eigenvalues of the automorphism* A^u_δ*; to each eigenvalue* λ_m*, there is an unique Jordan's block of dimension* $q_m + 1$ *and*

$$dim\ F^u_{(a_1,a)} = \sum_{m \in M} (q_m + 1).$$

Now, let u be the following germ at point (a_1, a)

$$u(t,x) = t^p \sum_{k=0}^{q} a_k(t,x)[\ln t]^k$$

where functions $a_k : \dot{D}_\omega \times \Omega \to \mathbf{C}$ are holomorphic, $a_q \not\equiv 0$. This germ u is of finite dimensional determination, dim $F^u_{(a_1,a)} = q + 1$ and $e^{2i\pi p}$ is the only eigenvalue of the automorphism A^u_δ. Moreover, it is easy to see that in all basis $(\psi_0, \ldots, \psi_q)$ of $F^u_{(a_1,a)}$, such that the matricial representation of A^u_δ in this basis is

$$\begin{pmatrix} e^{2i\pi p} & 1 & & \\ & \ddots & \ddots & \\ & & e^{2i\pi p} & 1 \\ & & & e^{2i\pi p} \end{pmatrix},$$

we have $\psi_0 = Ct^p a_q$ where$C \in \mathbf{C}^*$.

Before stating the main theorem, a new definition is introduced.

Definition 2.1 *Let* $p \in \mathbf{C}$ *such that* $\mathcal{R}e(p) \in [0,1[$ *and* $q \in \mathbf{N}$. $u \in h(p,q)$ *if there is a real* $\omega > 0$ *and a simply connected open neighbourhood* Ω *of the origin of* $\mathbf{C}^{n+1}$ *such that* $u \in \mathcal{H}(\mathcal{R}(\dot{D}_\omega) \times \Omega)$ *could be written under the form*

$$\sum_{k=0}^{q} t^p a_k(t,x)[\ln t]^k$$

where the a_k *are holomorphic on* $\dot{D}_\omega \times \Omega$ *and where* $a_q \in \mathcal{H}(D_\omega \times \Omega)$ *when* $p = 0$ *with* $D_\omega = \{t \in \mathbf{C};\ |t| < \omega\}$.

Theorem 2.2 *If the germs v_i, w_h are in a same $h(p,q)$ space, then the solution u of problem (1) could be written*

$$u(x) = \sum_{i=1}^{d} u_i(k_i(x), x)$$

with $u_i \in h(p,q)$ for $i = 1, \ldots, d$.

Remark 2.1 *Let $u \in h(p,q)$ and k a holomorphic function in a neighbourhood of the origin, then, for all holomorphic linear differential operator $a(x,D)$, we have*

$$a(x,D)u(k(x),x) = v(k(x),x)$$

where $v \in h(p,q)$. This results from the fact that the $h(p,q)$ spaces are invariant under differentiation.

This remark allows us to assume the germs $w_h \equiv 0$, in theorem 2.2.

We have to say that C. Wagschal [5] gave an analogous result in the case of an operator with simple characteristics (that is $m_i = 1$ for all i and then $d = m$). He defined the following $\tilde{h}(p,q)$ spaces.

Definition 2.2 *Let $p \in \mathbf{C}$ and $q \in \mathbf{N}$. $u \in \tilde{h}(p,q)$ if there is a real $\omega > 0$ and a simply connected open neighbourhood Ω of the origin of $\mathbf{C}^{n+1}$ such that $u \in \mathcal{H}(\mathcal{R}(\dot{D}_\omega) \times \Omega)$ could be written under the form*

$$\begin{cases} t^p P_q(x, [\ln t]) & \textit{if } p \textit{ is not a integer } < 0 \\ t^p P_{q-1}(x, [\ln t]) + P_q(x, [\ln t]) & \textit{if } p \textit{ is an integer } < 0 \textit{ and } q \geq 1 \\ P_0(x) & \textit{if } p \textit{ is an integer } < 0 \textit{ and if } q = 0. \end{cases}$$

where $P_i(x,\xi)$, $\xi \in \mathbf{C}$ are polynomial functions in ξ of degree $\leq i$, with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}^{n+1}$.

Then, he proved the following theorem [5]

Theorem 2.3 *If the germs v_i and w_h are in a same $\tilde{h}(p,q)$ space, then the solution u of problem (1) (in the case of an operator with simple characteristics) could be written*

$$u(x) = \sum_{i=1}^{d} u_i(k_i(x), x)$$

with $u_i \in \tilde{h}(p,q)$ for $i = 1, \ldots, d$.

Also, this theorem gives a result on the order of the polar singularity of the solution. This type of result could not be obtained for an operator with multiple characteristics of constant multiplicity. Indeed, Y. Hamada [2] gave the following very simple counterexample

$$\begin{cases} [D_0^2 - D_1]u(x) = 0, \\ \\ u(x) = \dfrac{1}{x_1} \quad \text{ for } x_0 = 0, \\ \\ D_0 u(x) = 0 \quad \text{ for } x_0 = 0. \end{cases}$$

The solution is

$$u(x) = \frac{1}{x_1} + \frac{1}{x_1}\sum_{n=1}^{\infty}(-1)^n \frac{n!}{(2n)!}\Big(\frac{x_0^2}{x_1}\Big)^n;$$

then the solution u have only essential singularities on the double characteristic hypersurface

$$K = \{x;\ x_1 = 0\}.$$

3 Recall on the monodromy

We come back to the problem (1)

$$\begin{cases} a(x,D)u(x) & = \displaystyle\sum_{i=1}^{d} v_i(t,x)_{\big|t=k_i(x)}, \\ \\ D_0^h u(x) & = w_h(t,x)_{\big|t=x_1} \quad \text{ for } x_0 = 0,\ 0 \le h < m. \end{cases}$$

The solution of this problem could be written under the form

$$u(x) = \sum_{i=1}^{d} u_i(t,x)_{\big|t=k_i(x)} = \sum_{i=1}^{d}[D_0^{m-m_i} D_t^{m_i} U_i(t,x)]_{\big|t=k_i(x)}$$

where U_i is a germ at point (a_1, a) which has a holomorphic extension on $\mathcal{R}(\dot{D}_\omega)\times\Omega'$. $U = (U_1, \ldots, U_d)$ is the solution of an integro-differential problem

$$\begin{cases} (D_0^m - A_m(x,D))U(t,x) = \displaystyle\sum_{l\in\mathcal{L}} A_l^m(x,D)D_t^{-l}U(t,x) + y_m(t,x), \\ \\ (D_0^h - A_h(x,D))U(t,x) \\ \qquad\qquad = \displaystyle\sum_{l\in\mathcal{L}} A_l^h(x,D)D_t^{-l}U(t,x) + y_h(t,x) \text{ for } x_0 = 0,\ 0 \le h < m \end{cases} \tag{3}$$

with $\mathcal{L}$ a finite subset of $\mathbf{Z}$,

$$y_m(t,x) = (a_1(x) D_t^{-m} v_1(t,x), \dots, a_d(x) D_t^{-m} v_d(t,x))$$

where a_i are holomorphic functions on Ω and, for $0 \leq h \leq m-1$,

$$y_h(t,x) = \left(\sum_{k=0}^{m-1} P_{1,k,h}(x', D_t^{-1}) w_k(t,x), \dots, \sum_{k=0}^{m-1} P_{d,k,h}(x', D_t^{-1}) w_k(t,x) \right)$$

where $P_{i,j,h}(x', \xi)$ is polynomial function in $\xi \in \mathbf{C}$ with holomorphic coefficients in the variable x'. This kind of integro-differential problem was studied in [3, 4]. Section 4. of [1] shows that if the datas of problem (3) are of finite dimensional determination, then

$$F^U_{(a_1,a)} \subset F$$

where F is a finite dimensional vector space invariant under analytic continuation. F is spanned by the generators' system

$$\Big(S(Q_{\delta^{\pm 1}} S(\theta_j)), S(\theta_j) \Big)_{1 \leq j \leq q}, \tag{4}$$

with the following notations

- $(\theta_j)_{1 \leq j \leq q}$ is a basis of $F^y_{(a_1,a)}$, where $y = (y_0, \dots, y_m)$,
- $S(y)$ is the solution of problem (3),
- $(Q_{\delta^{\pm 1}} u)_h = \sum_{l \in \mathcal{L}^+} A_l^h(x,D) P^l_{\delta^{\pm 1}}(u)$, $\mathcal{L}^+ = \{l \in \mathcal{L};\ l > 0\}$ and

$$P^l_{\delta^{\pm 1}}(u) = \left(D_t^{-l} u \right)_{\delta^{\pm 1}} - D_t^{-l}(u_{\delta^{\pm 1}}).$$

We can show that $P^l_{\delta^{\pm 1}}(u)$ is holomorphic in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x^{n+1}$, that $P^l_{\delta^{\pm 1}}(u) = 0$ if u is holomorphic in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x^{n+1}$ [1]. Moreover, if y is holomorphic in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x^{n+1}$, then so is $S(y)$ [4]. Hence $S(Q_{\delta^{\pm 1}} S(\theta_j))$ is holomorphic in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x^{n+1}$.

These results allow us to show that a matricial representation of the automorphism $\theta \mapsto \theta_\gamma$, where $\gamma \in \Gamma_{(a_1,a)}(\dot{D}_\omega \times \Omega)$, in the generators' system (4) could be written

$$\begin{pmatrix} Id & B \\ 0 & M \end{pmatrix}$$

where M is the matricial representation of A^y_γ in the basis $(\theta_j)_j$.

At last, for all $i = 1, \dots, d$, thanks to the results of section 3. of [1], we know that there is a vector space F_i of finite dimension which is invariant under analytic continuation such that

$$F^{u_i}_{(a_1,a)} \subset F_i.$$

There is a generators' system of F_i such that a matricial representation of the automorphism $\theta \mapsto \theta_\gamma$ in this generators' system is

$$\begin{pmatrix} Id & B \\ 0 & M \end{pmatrix}.$$

Moreover, in section 5. of [1] the following fact was shown. If $1 \notin \sigma_\delta(y)$, then there is a generators' system of F_i such that a matricial representation of the automorphism $\theta \mapsto \theta_\gamma$ in this generators' system is M.

4 Proof of theorem 2.2

By linearity, we can assume $v_1 \in h(p,q)$, $v_1 \neq 0$ and $v_2, \dots, v_d = 0$.

The following problem is studied

$$\begin{cases} a(x,D)u(x) & = v_1(t,x)_{|t=k_1(x)}, \\ \\ D_0^h u(x) & = 0 \quad \text{for } x_0 = 0,\ 0 \leq h < m. \end{cases}$$

Thanks to the previous section, the solution could be written under the form

$$u(x) = \sum_{i=1}^{d} u_i(t,x)_{|t=k_i(x)} = \sum_{i=1}^{d} [D_0^{m-m_i} D_t^{m_i} U_i(t,x)]_{|t=k_i(x)}$$

where U_i is a germ at point (a_1, a) which has a holomorphic extension on $\mathcal{R}(\dot{D}_\omega) \times \Omega'$. $U = (U_1, \dots, U_d)$ is solution of an integro-differential problem

$$\begin{cases} (D_0^m - A_m(x,D))U(t,x) = \sum_{l \in \mathcal{L}} A_l^m(x,D) D_t^{-l} U(t,x) + y_m(t,x), \\ \\ (D_0^h - A_h(x,D))U(t,x) \\ \qquad = \sum_{l \in \mathcal{L}} A_l^h(x,D) D_t^{-l} U(t,x) \text{ for } x_0 = 0,\ 0 \leq h < m \end{cases}$$

with

$$y_m(t,x) = (a(x) D_t^{-m} v_1(t,x), 0, \dots, 0)$$

where a is a holomorphic function on Ω.

With help of section 4. of [1], we know that $F^U_{(a_1,a)} \subset F$ where F is a finite dimensional vector space invariant under analytic continuation. A generators' system of F is given by

$$\Big(S(Q_{\delta^{\pm 1}}S(\beta_j)), S(P^m_{\delta^{\pm 1}}(D^m_t\beta_j)), S(\beta_j)\Big)_{0\leq j\leq k}$$

where $\beta_j = (a(x)D_t^{-m}\theta_j, 0\ldots, 0)$ and $(\theta_j)_j$ is a basis of $F^{v_1}_{(a_1,a)}$.

Remark 4.1 *We can show that F is spanned by*

$$(S(Q_\delta S(\beta_j)), S(P^m_\delta(D^m_t\beta_j)), S(\beta_j))_{0\leq j\leq k}. \tag{5}$$

Indeed, thanks to lemma 3.4. of [1], we get

$$P^l_e(u) = P^l_{\delta^{-1}\delta}(u) = P^l_{\delta^{-1}}(u) + P^l_\delta(u_{\delta^{-1}});$$

but $P^l_e(u) = 0$, so $P^l_{\delta^{-1}}(u) = -P^l_\delta(u_{\delta^{-1}})$. Then we have

$$P^m_{\delta^{-1}}(D^m_t\beta_j) = -P^m_\delta((D^m_t\beta_j)_{\delta^{-1}}).$$

But the vector space spanned by $(D^m_t\beta_j)_{0\leq j\leq k}$ is invariant under analytic continuation (because $D^m_t\beta_j = (a\theta_j, 0\ldots, 0)$), so $S(P^m_{\delta^{-1}}(D^m_t\beta_j))$ lies in the vector space spanned by

$$(S(Q_{\delta^{\pm 1}}S(\beta_j)), S(P^m_\delta(D^m_t\beta_j)), S(\beta_j))_{0\leq j\leq k}.$$

Now, from definition of operators Q_γ and from the fact that $S(u)_\gamma = S(u_\gamma) + S(Q_\gamma S(u))$, we have

$$\begin{array}{rcl} Q_{\delta^{-1}}S(\beta_j) & = & -Q_\delta((S(\beta_j))_{\delta^{-1}}) \\ & = & -Q_\delta S((\beta_j)_{\delta^{-1}}) - Q_\delta(S(Q_{\delta^{-1}}(\beta_j))) \\ & = & -Q_\delta S((\beta_j)_{\delta^{-1}}) \end{array}$$

because $S(Q_{\delta^{-1}}S(\beta_j)) \in \mathcal{H}(D_\omega \times \Omega')$. Now, since the vector space spanned by $(P^m_\delta(D^m_t\beta_j), \beta_j)_{0\leq j\leq k}$ is invariant under analytic continuation and since $P^m_\delta(D^m_t\beta_j)$ is holomorphic on $D_\omega \times \Omega'$, $S(Q_\delta S((\beta_j)_{\delta^{-1}}))$ lies in the vector space spanned by

$$(S(Q_\delta S(\beta_j)), S(P^m_\delta(D^m_t\beta_j)), S(\beta_j))_{0\leq j\leq k}.$$

Hence, F is spanned by this generators' system.

First, we look to the case $p \neq 0$.
There is a basis of $F^{v_1}_{(a_1,a)}$ such that the matricial representation of $A^{v_1}_\delta$ in this basis is

$$M = \begin{pmatrix} e^{2i\pi p} & 1 & & \\ & \ddots & \ddots & \\ & & e^{2i\pi p} & 1 \\ & & & e^{2i\pi p} \end{pmatrix}.$$

Moreover, M is a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ in a generators' system of F_i because $1 \notin \sigma(v_1)$. Therefore, we know that the solution of problem (1) is of finite dimensional determination and could be written under the form

$$u(x) = \sum_{i=1}^{d} u_i(k_i(x), x)$$

where $F^{u_i}_{(a_1,a)} \subset F_i$

The end of the proof use corollary 3.1. of [1] which is recalled.

Corollary 4.1 *Let F be a finite dimensional vector space, $A \in \mathcal{L}(F)$ a homomorphism whose a matricial representation in a generators' system is given by a matrix M. Then, we have*

$$\sigma(A) \subset \sigma(M)$$

where $\sigma(A)$ is the set of eigenvalues of A and $\sigma(M)$ is the set of eigenvalues of M.

Hence, with help of theorem 2.1 and corollary 4.1, we deduce that $u_i \in h(p,q)$ for $p \neq 0$.

Now, consider the case $p = 0$.

We have

$$v_1(t,x) = \sum_{k=0}^{q} a_k(t,x)[\ln t]^k$$

with $a_q \in \mathcal{H}(D_\omega \times \Omega)$. By linearity, two cases will be studied:

1. $v_1(t,x) = a_k(t,x)[\ln t]^k$ with $0 \leq k \leq q-1$,
2. $v_1(t,x) = a_q(t,x)[\ln t]^q$.

First case – Let $(\theta_0, \ldots, \theta_k)$ be a basis of $F^{v_1}_{(a_1,a)}$ such that the matricial representation of the automorphism $A^{v_1}_\delta$, in this basis, is

$$M_{k+1} = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

We recall that $\theta_0 = K a_k$ où $K \in \mathbf{C}^*$.

A matricial representation of $\theta \mapsto \theta_\delta$ in the generators' system (5) is given by

$$\begin{pmatrix} Id & B \\ 0 & M_{k+1} \end{pmatrix} \tag{6}$$

where

$$B = \begin{pmatrix} Id \\ Id \end{pmatrix}.$$

Indeed, we have

- $S(Q_\delta S(\beta_j))_\delta = S(Q_\delta S(\beta_j))$ since $S(Q_\delta S(\beta_j)) \in \mathcal{H}(D_\omega \times \Omega')$,
- $S(P_\delta^m(D_t^m \beta_j))_\delta = S(P_\delta^m(D_t^m \beta_j))$ since $S(P_\delta^m(D_t^m \beta_j)) \in \mathcal{H}(D_\omega \times \Omega')$,
- $S(\beta_j)_\delta = S(\beta_{j\delta}) + S(Q_\delta S(\beta_j))$ and $\beta_{j\delta} = P_\delta^m(D_t^m \beta_j) + \sum_{i=1}^{k+1} \lambda_{ij}^\delta \beta_i$ where $(\lambda_{ij}^\delta)_{ij} = M_{k+1}$.

The matrix (6) could be written in an another basis, under the following Jordan's form

$$\begin{pmatrix} Id & 0 \\ 0 & M_{k+2} \end{pmatrix}; \tag{7}$$

the change of basis is

$$\begin{pmatrix} C & D \\ 0 & Id_{k+1} \end{pmatrix}$$

where Id_{k+1} is the identity matrix of dimension $k+1$. We denote the generators' system of the Jordan's block M_{k+2} by $(\psi_0, \ldots, \psi_{k+1})$. With the help of the matrix of change of basis, we can see that ψ_0 is holomorphic on $D_\omega \times \Omega'$. The following lemma is fundamental to end the proof of theorem 2.2.

Lemma 4.1 *Assume that F is spanned by a generators' system such that a matricial representation $\theta \mapsto \theta_\delta$, in this generators' system, could be written*

$$\begin{pmatrix} Id & 0 \\ 0 & M \end{pmatrix}$$

where M is a Jordan's matrix and 1 is an eigenvalues of M. Then, the solution of problem (1) could be written under the form

$$\sum_{i=1}^{d} u_i(t,x)_{\big| t=k_i(x)}$$

with $F_{(a_1,a)}^{u_i} \subset F_i$ where F_i is a finite dimensional vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ of F_i in a generators' system is M.

Proof – We denote by $M = diag(J_{\lambda_1}, \ldots, J_{\lambda_n})$ where J_{λ_j} is a Jordan's matrix with unique eigenvalues λ_j. We assume $\lambda_1 = 1$. Also, we denote $(\varphi_k, \psi_l)_{kl}$, where $\varphi_k \in \mathcal{H}(\dot{D}_\omega \times \Omega')$, $\psi_l \in \mathcal{H}(\mathcal{R}(\dot{D}_\omega) \times \Omega')$, a generators' system of F such that a maticial representation of $\theta \mapsto \theta_\delta$ in this generators' system is

$$\begin{pmatrix} Id & 0 \\ 0 & M \end{pmatrix}.$$

We have, for all $i = 1, \dots, d$,

$$U_i(t,x) = \sum_k \lambda_k \varphi_k^i + \sum_l \mu_l \psi_l^i.$$

Denote $\varphi = (\sum_k \lambda_k \varphi_k^i)_{1 \le i \le d}$ and $(\psi_1, \dots, \psi_{l'})$ the generators' system associated to the Jordan's block J_1. If, for $l = 1, \dots, l'$, $\mu_l = 0$, then we denote $\tilde{\psi}_1 = \varphi$ and we add to $\tilde{\psi}_1$ some vectors $\tilde{\psi}_2, \dots, \tilde{\psi}_{l'}$ to form a vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$, in the generators' system, $(\tilde{\psi}_1, \dots, \tilde{\psi}_{l'})$ is J_1. Denote, for $l > l'$, $\tilde{\psi}_l = \psi_l$. Then, we have

$$U_i = \sum_l \mu_l' \tilde{\psi}_l^i.$$

Thus $F_{(a_1,a)}^{U_i} \subset \tilde{F}_i$ where $\tilde{F}_i$ is a vector space invariant under analytic continuation such that a matricial representaton of the automorphism $\theta \mapsto \theta_\delta$, in the generators' system $(\tilde{\psi}_l)$, is M. Hence, we have $F_{(a_1,a)}^{u_i} \subset F_i$ where F_i is a vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ is M.

Now, denote $l_0 = \max\{l = 1, \dots, l';\ \mu_l \neq 0\}$. So, introducing

$$\begin{cases} \tilde{\psi}_l = \psi_l \text{ if } l > l', \\ \tilde{\psi}_l = \psi_l \text{ if } l < l_0, \\ \tilde{\psi}_{l_0} = \psi_{l_0} + \dfrac{1}{\mu_{l_0}} \varphi \end{cases}$$

and adding to $(\tilde{\psi}_1, \dots, \tilde{\psi}_{l_0})$ some vectors $\tilde{\psi}_{l_0+1}, \dots, \tilde{\psi}_{l'}$ to form a vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ in the generators' system $(\tilde{\psi}_1, \dots, \tilde{\psi}_{l'})$ is J_1. Then, we have

$$U_i' = \sum_l \mu_l \tilde{\psi}_l^i.$$

So, we have $F_{(a_1,a)}^{U_i} \subset \tilde{F}_i$ where $\tilde{F}_i$ is a vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$, in the generators' system $(\tilde{\psi}_l)$, is M. Hence, we have $F_{(a_1,a)}^{u_i} \subset F_i$ where F_i is a vector space invariant under analytic continuation such that a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ is M. This is end the proof of the lemma. ■

The matrix (7) is a Jordan's matrix; thereby, with help of lemma 4.1, we could write

$$u(x) = \sum_{i=1}^d u_i(k_i(x), x)$$

with $F^{u_i}_{(a_1,a)} \subset F_i$ where F_i is a vector space invariant under analytic continuation and such that a matricial representation in a generators' system is given by M_{k+2}. Hence, with help of theorem 2.1 and corollary 4.1, we have

$$u_i(t,x) = \sum_{j=0}^{k+1} a_j(t,x)[\ln t]^j$$

where $a_j \in \mathcal{H}(\dot{D}_\omega \times \Omega')$ and $a_k = K_i \psi_0^i \in \mathcal{H}(D_\omega \times \Omega')$ (eventually null). Hence $u_i \in h(0, k+1)$.

Second case – As previously, a generators' system of the vector space F is given by

$$\Big(S(Q_\delta S(\beta_l)), S(P^m_\delta(D^m_t \beta_l)), S(\beta_l)\Big)_{0 \le l \le q}$$

where $\beta_l = (a(x) D_t^{-m} \theta_l, 0 \ldots, 0)$ with $a \in \mathcal{H}(\Omega)$.
We have to remark that, since $\theta_0 \in \mathcal{H}(D_\omega \times \Omega)$, we have $S(\beta_0) \in \mathcal{H}(D_\omega \times \Omega')$ and then $S(Q_\delta S(\beta_0)) = S(P^m_\delta(D^m_t \beta_0)) = 0$.
Then, a generators' system of F is given by

$$\Big(S(Q_\delta S(\beta_l)), S(P^m_\delta(D^m_t \beta_l)), S(\beta_0), S(\beta_l)\Big)_{1 \le l \le q}$$

and a matricial representation of the automorphism $\theta \mapsto \theta_\delta$ in this generators' system is given by

$$\begin{pmatrix} Id & B' \\ 0 & M_{q+1} \end{pmatrix} \tag{8}$$

where

$$B' = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \end{pmatrix}.$$

This follows from the same fact as above and $S(\beta_0) \in \mathcal{H}(D_\omega \times \Omega')$.
Now, (8) could be written in an another basis under the form

$$\begin{pmatrix} Id & 0 \\ 0 & M_{q+1} \end{pmatrix}. \tag{9}$$

The matrix of change of basis is

$$\begin{pmatrix} C & D \\ E & Id_{q+1} \end{pmatrix} \tag{10}$$

where

$$E = \begin{pmatrix} a_1 & \dots & a_{2q} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Let $(\varphi_1, \dots, \varphi_{2q}, \psi_0, \dots, \psi_q)$ the new generators' system obtained by the change of basis. Then, with help of the matrix's form (10), for all $j = 1, \dots, 2q$, φ_j is holomorphic on $D_\omega \times \Omega'$ and so is ψ_0.
The matrix (9) is a Jordan's matrix. Thus, with help of lemma 4.1, we could write

$$u(x) = \sum_{i=1}^{d} u_i(k_i(x), x)$$

with $F^{u_i}_{(a_1,a)} \subset F_i$ where F_i is a vector space invariant under analytic continuation and such that a matricial representation in a generators' system is given by M_{q+1}. So, with help of theorem 2.1 and corollary 4.1, we have

$$u_i(t, x) = \sum_{j=0}^{k} a_j(t, x)[\ln t]^j$$

where $a_j \in \mathcal{H}(\dot{D}_\omega \times \Omega')$ and $a_q = K_i \psi_0^i \in \mathcal{H}(D_\omega \times \Omega')$. Hence $u_i \in h(0, q)$. This ends the proof of theorem 2.2.

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