Elliptic Theory on Manifolds with Nonisolated Singularities

II. Products in Elliptic Theory on Manifolds with Edges

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Abstract

Exterior tensor products of elliptic operators on smooth manifolds and manifolds with conical singularities are used to obtain examples of elliptic operators on manifolds with edges that do not admit well-posed edge boundary and coboundary conditions.

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Introduction

A well-known tool of elliptic theory is the so-called exterior tensor product of operators, introduced by Atiyah and Singer in [1]. Recall that the exterior tensor product $D_1 \# D_2$ of two elliptic operators D_1 and D_2 on smooth manifolds M_1 and M_2 is some elliptic operator on the product $M_1 \times M_2$. If both manifolds are closed, then the index of the tensor product is equal to the product of indices of the factors

$$\operatorname{ind} D_1 \# D_2 = \operatorname{ind} D_1 \operatorname{ind} D_2. \tag{0.1}$$

Exterior tensor products and formulas of this type play a fundamental role in the index theory of elliptic operators: the first proof of the Atiyah–Singer index theorem (e.g., see [2]) used formula (0.1). A more general formula is valid when one considers a locally trivial fiber bundle, instead of a direct product of manifolds, see [3]. This formula was used in the K-theoretic proof of the index theorem.

The aim of the present paper is to construct a certain product of elliptic operators defined on manifolds with isolated singularities and operators defined on smooth manifolds. We show that as a result of this product we obtain a very interesting class of operators on manifolds with edges. Indeed, this class contains some operators that require no (co)boundary conditions along the edge to obtain the Fredholm property. On the other hand, this class contains some operators for which the obstruction to the existence of (co)boundary edge conditions (see [7]) is nonzero. We also give explicit examples of operators that have no elliptic edge problems.

This paper is organized as follows. First we briefly recall the exterior tensor product. We then pass to the exterior tensor product of an elliptic operator on a smooth manifold by an operator defined on manifold with an isolated conical singularity. We prove that the resulting edge-degenerate operator does not require boundary conditions along the edge. Moreover, we employ exterior tensor products to construct another class of edge-degenerate operators. For these operators we compute the obstruction to the existence of an elliptic edge boundary value problem. We finally give some examples where the obstruction does not vanish.

1 Exterior tensor product of elliptic operators

Let M_1 and M_2 be two smooth manifolds (possibly, with boundary). Consider two elliptic differential operators of order m

$$D_1 : C^\infty(M_1, E_1) \longrightarrow C^\infty(M_1, F_1), \quad D_2 : C^\infty(M_2, E_2) \longrightarrow C^\infty(M_2, F_2). \tag{1.1}$$

The operators act on sections of vector bundles E_1, E_2, F_1, F_2 over the corresponding manifolds.

The *exterior tensor product* $D_1 \# D_2$ of operators D_1 and D_2, is an operator on the Cartesian product $M = M_1 \times M_2$ defined as

$$D_1 \# D_2 = \begin{pmatrix} D_1 \otimes 1_{E_2} & 1_{F_1} \otimes D_2^* \\ 1_{E_1} \otimes D_2 & -D_1^* \otimes 1_{F_2} \end{pmatrix} : \begin{array}{c} C^\infty(M, E_1 \otimes E_2) \\ \oplus \\ C^\infty(M, F_1 \otimes F_2) \end{array} \longrightarrow \begin{array}{c} C^\infty(M, F_1 \otimes E_2) \\ \oplus \\ C^\infty(M, E_1 \otimes F_2). \end{array} \tag{1.2}$$

Here for brevity the pull-backs to M of vector bundles from the factors are denoted by the same letters as the initial bundles. The formal adjoint operators are taken with respect to some Hermitian metrics in vector bundles and some smooth measures on the manifolds.

By $A \otimes 1_E$ we denote the operator on the product of two manifolds that corresponds to some operator A on one factor and a vector bundle E on the other factor.

It follows from the relation

$$(D_1 \# D_2)^*(D_1 \# D_2) = \begin{pmatrix} D_1^* D_1 \otimes 1_{E_2} + 1_{E_1} \otimes D_2^* D_2 & 0 \\ 0 & D_1 D_1^* \otimes 1_{F_2} + 1_{F_1} \otimes D_2 D_2^* \end{pmatrix} \tag{1.3}$$

that the tensor product of elliptic operators is an elliptic operator. For brevity here and below by a tensor product of operators on different manifolds we mean the exterior tensor product.

From the topological point of view, one can say that the expression (1.2) on the level of principal symbols of the corresponding elliptic operators defines a product

$$K(T^*M_1) \times K(T^*M_2) \longrightarrow K(T^*(M_1 \times M_2))$$

of the K-groups of the cotangent bundles of the manifolds. More precisely, the difference constructions of elliptic symbols $\sigma(D_1)$ and $\sigma(D_2)$ are mapped to the difference construction of the product $\sigma(D_1)\#\sigma(D_2)$.

Atiyah, Patodi and Singer showed in [4] that the difference construction of the symbol of an elliptic *self-adjoint* operator is an element of the odd K-group $K^1(T^*M)$ of the cotangent bundle of a smooth manifold M. On the other hand, it is obvious that if the first factor D_1 of the tensor product is self-adjoint then the result $D_1 \# D_2$ turns out to be self-adjoint as well. Thus, one sees that in this case the tensor product also induces a product in K-theory:

$$K^1(T^*M_1) \times K(T^*M_2) \longrightarrow K^1(T^*(M_1 \times M_2)).$$

The remaining third product

$$K^1(T^*M_1) \times K^1(T^*M_2) \longrightarrow K(T^*(M_1 \times M_2))$$

of two odd K-groups can also be defined in terms of elliptic operators. Namely, for a pair D_1, D_2 of elliptic self-adjoint operators the tensor product is defined by the formula

$$D_1 \# D_2 = D_1 \otimes 1_{E_2} + i\,(1_{E_1} \otimes D_2) : C^\infty(M, E_1 \otimes E_2) \longrightarrow C^\infty(M, E_1 \otimes E_2). \tag{1.4}$$

Of course, this operator is not self-adjoint.

2 Example 1. An operator on a manifold with edge that does not require edge conditions

Let us apply the tensor product when manifolds have singularities. We consider the simplest case.

Assume that the smooth manifold M_1 is closed and operator D_1 on it has smooth coefficients. Moreover, let M_2 be a manifold with boundary and assume D_2 on the interior $\overset{\circ}{M}_2$ to have a conical degeneracy (see [5]), i.e. it has smooth coefficients in the interior, while in a certain neighborhood of the boundary it has a decomposition of the form

$$D_2|_{U_{\partial M_2}} = r^{-m} \sum_{0\le k\le m} a_k\left(\omega, r, -i\frac{\partial}{\partial\omega}\right)\left(ir\frac{\partial}{\partial r}\right)^{m-k},$$

where r,ω denote the normal and the tangential variables at the boundary, while the coefficients $a_k\left(\omega, r, -i\frac{\partial}{\partial\omega}\right)$ are smooth families with parameter r of differential operators of order $0\le k\le m$ on the boundary of M_2. The principal symbol of the corresponding coefficient will be denoted by $a_k(\omega, r, q)$. In this case D_2 is a continuous operator in the scale of weighted Sobolev spaces on M_2:

$$D_2 : H^{s,\gamma}(M_2) \longrightarrow H^{s-m,\gamma-m}(M_2). \tag{2.1}$$

Let us assume that both D_1 and D_2 are elliptic. Recall that for conically degenerate operators the ellipticity condition amounts to two requirements:

a) interior ellipticity: for $r\neq 0$ the principal symbol of the operator must be invertible, and for $r=0$ the symbol

$$\sum_{0\le k\le m} a_k(\omega, 0, q)\,\rho^{m-k}$$

must be invertible for all $\rho^2+q^2\neq 0$ as well;[1]

b) the *conormal symbol*

$$\sigma_c(D_2)(p) = \sum_{0\le k\le m} a_k\left(\omega, 0, -i\frac{\partial}{\partial\omega}\right)p^{m-k}$$

must be invertible as operators

$$\sigma_c(D_2)(p) : H^s(\partial M_2) \longrightarrow H^{s-m}(\partial M_2)$$

for all values of the parameter p on the weight line

$$L_\gamma = \{p\in\mathbb{C} \mid \operatorname{Im} p = \gamma\}.$$

(The conormal symbol is an operator family on the boundary, elliptic with parameter p.)

It is known that an elliptic operator D_2 defines a Fredholm operator in weighted Sobolev spaces (2.1); e.g., see [7].

[1] These two conditions can be combined if one considers the symbol of the operator on the so called compressed cotangent bundle $T^*\overline{M}_2$ of the manifold with conical singularities, e.g., see [6]

Let us now consider the tensor product of D_1 and D_2. Near the singularity, i.e. for small r, the product has the form

$$D_1 \# D_2\big|_{M_1 \times U_{\partial M_2}} = r^{-m} \begin{pmatrix} r^m D_1 \otimes 1_{E_2} & 1_{F_1} \otimes r^m D_2^* \\ 1_{E_1} \otimes r^m D_2 & -r^m D_1^* \otimes 1_{F_2} \end{pmatrix}. \tag{2.2}$$

Here, just as in the preceding, D^* is the formally adjoint operator. For manifolds with conical singularities the adjoint is taken with respect to the measure $r^{-1} d\omega dx dr$.

Let us consider $\overset{\circ}{M}_2$ as the interior of a manifold

$$\overline{M}_2 = M_2 / \, \partial M_2$$

with conical singularity obtained under the identification of all points of the boundary. Now geometrically the product (2.2) corresponds to the manifold

$$\mathbb{W} = M_1 \times \overline{M}_2$$

with edge M_1.

From the analytical point of view, (2.2) also corresponds to this singular manifold, since the corresponding operator is *edge-degenerate*, i.e. it has a decomposition of the form (cf. [8])

$$P = r^{-m} \sum_{k+|l| \le m} b_{k,l} \left(r, x, \omega, -i \frac{\partial}{\partial \omega} \right) \left(ir \frac{\partial}{\partial r} \right)^k \left(-ir \frac{\partial}{\partial x} \right)^l, \tag{2.3}$$

where the multiindex l has length equal to the dimension of M_1, while the coefficients

$$b_{k,l} \left(r, x, \omega, -i \frac{\partial}{\partial \omega} \right)$$

denote smooth families of differential operators of order $m - k - |l|$ on ∂M_2 with smooth coefficients. The operators smoothly depend on parameters r, x up to $r = 0$.

It is well known (see [5]) that the ellipticity condition for edge-degenerate operators consists of two parts:

a) interior ellipticity, i.e. the invertibility of the principal symbol for $r \neq 0$ and the invertibility of the symbol

$$\sum_{k+|l| \le m} b_{k,l} \left(0, x, \omega, q \right) p^k \xi^l$$

for $p^2 + q^2 + \xi^2 \neq 0$;

b) the invertibility of the so called *edge-symbol* of the operator. For operator P as in (2.3), the *edge-symbol* is defined as

$$\sigma_\Lambda(P)(x, \xi) = r^{-m} \sum_{k+|l| \le m} b_{k,l} \left(0, x, \omega, -i \frac{\partial}{\partial \omega} \right) \left(ir \frac{\partial}{\partial r} \right)^k (r\xi)^l .$$

It is a family of operators parametrized by the cosphere bundle S^*M_1 of the edge. More precisely, each operator of the family acts in special weighted spaces:

$$\sigma_\Lambda(P)(x,\xi) : \mathcal{K}^{s,\gamma}(K_{\partial M_2}) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_{\partial M_2})$$

on the infinite cone (here K_Ω denotes the infinite cone

$$K_\Omega \simeq \left\{\Omega \times \mathbb{R}_+\right\} / \left\{\Omega \times \{0\}\right\} \tag{2.4}$$

with base Ω).

Let us recall the definition of the spaces $\mathcal{K}^{s,\gamma}$ (see [5]).

For a smooth compact manifold Ω of dimension $n-1$ the space $\mathcal{K}^{s,\gamma}(K_\Omega)$ on the infinite cone K_Ω is defined as the completion of $C_0^\infty(\Omega \times \mathbb{R}_+)$ with respect to the norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(K_\Omega)} = \|\varphi_1 u\|_{H^{s,\gamma}(K_\Omega)} + \|\varphi_2 u\|_{H^s(K_\Omega \cap \{r>R\})}, \tag{2.5}$$

where $\varphi_1(r) + \varphi_2(r) = 1$ is a partition of unity on $\mathbb{R}_+$ such that φ_1 is zero at infinity and φ_2 is zero near the origin, while $H^s(K_\Omega \cap \{r > R\})$ is the "usual" Sobolev space that is obtained if we cover Ω by local charts diffeomorphic to some domains on the sphere $\mathbb{S}^{n-1}$ and interpret (r,ω) as polar coordinates in $\mathbb{R}^n$ and rewrite in these coordinates the usual Sobolev norm. Possibly a simpler description of this space for nonnegative integer s can be given by an equivalent norm:

$$\|u\|_{H^s(K_\Omega \cap \{r>R\})} = \int\limits_{K_\Omega} r^{n-1-2s} \left\{ r^{2s}|u|^2 + \left| \left(r\frac{\partial}{\partial r}\right)^s u \right|^2 + \left((-\triangle_\Omega)^s u, u\right) \right\} dr\, d\omega,$$

$u \in C_0^\infty(K_\Omega \cap \{r > R\})$, where $\triangle_\Omega$ is the Beltrami–Laplace operator on Ω.

Under the above-mentioned ellipticity conditions, the operator (2.3) is Fredholm in the wedge Sobolev spaces

$$P : W^{s,\gamma}(\mathbb{W}) \longrightarrow W^{s-m,\gamma-m}(\mathbb{W}).$$

(The definition of these spaces, as well as the proof of the Fredholm property, can be found, e.g., in [5, 7]. From now on we assume that $\gamma = m/2$.)

Let us show that the ellipticity condition holds for the tensor product of our operators. The edge symbol of the tensor product (2.2) is given by

$$\sigma_\Lambda(D_1 \# D_2)(x,\xi) = r^{-m} \begin{pmatrix} \sigma(D_1)(x, r\xi) & \sigma_c(D_2^*)\left(ir\frac{\partial}{\partial r}\right) \\ \sigma_c(D_2)\left(ir\frac{\partial}{\partial r}\right) & -\sigma^*(D_1)(x, r\xi). \end{pmatrix}$$

(We write A instead of $A \otimes 1$ etc. for brevity.) The conormal symbol of this family is equal to

$$\sigma_c\left(\sigma_\Lambda(D)\right) = \begin{pmatrix} 0 & \sigma_c(D_2^*)\,(p) \\ \sigma_c(D_2)\,(p) & 0 \end{pmatrix}.$$

It is well known that if a conically degenerate operator D_2 of order m is Fredholm for a weight γ, then the adjoint operator is Fredholm for the weight $m-\gamma$; for $\gamma = m/2$ the two weights coincide. Under this condition, the conormal symbol of the edge symbol is invertible, so that the edge symbol is a family of Fredholm operators. To prove that this family is actually invertible, we shall show that the kernel of each operator of the family, as well as the kernel of the adjoint operator, is trivial.

To this end, just as in Section 1, we consider the product

$$T = \sigma_\Lambda^*(D_1 \# D_2)\sigma_\Lambda(D_1 \# D_2)$$

(where the adjoint operator is taken with respect to the inner product in the space $\mathcal{K}^{0,0}$) and prove that the kernel $\ker T = \ker \sigma_\Lambda(D_1\#D_2)$ is trivial. A straightforward computation shows that this product, modulo terms of order $2m-1$, has the form

$$T \simeq r^{-2m}\begin{pmatrix} \sigma^*(D_1)\sigma(D_1)(x,r\xi) + \sigma_c^*(D_2)\sigma_c(D_2)\left(ir\frac{\partial}{\partial r}\right) & 0 \\ 0 & \sigma(D_1)\sigma^*(D_1)(x,r\xi) + \sigma_c(D_2)\sigma_c^*(D_2)\left(ir\frac{\partial}{\partial r}\right). \end{pmatrix} \tag{2.6}$$

(Lower-order terms arise from commutations with the weight.) Moreover, the equality is exact for $r < \varepsilon$, where $\varepsilon > 0$ is sufficiently small, since the weight is equal to 1 in this region. It follows that if a vector function $u(r,\omega)$ lies in the kernel of T, then $u = 0$ for $r < \varepsilon$. Indeed, suppose the opposite: $Tu = 0$, but $u \not\equiv 0$ for $r < \varepsilon$. Consider a smooth partition of unity $1 = \chi_1^2(r) + \chi_2^2(r)$, where χ_1 and χ_2 are real, $\chi_1 = 0$ for $r > \varepsilon$, and $\chi_1 > 0$ and $r < \varepsilon$. Then we arrive at a contradiction:

$$\begin{aligned} 0 = (u, Tu) &= (\chi_1 u, T\chi_1 u) + (\chi_2 u, T\chi_2 u) \geq (\chi_1 u, T\chi_1 u) \\ &\geq (\chi_1 u, \operatorname{diag}\{\sigma^*(D_1)\sigma(D_1)(x,r\xi), \sigma(D_1)\sigma^*(D_1)(x,r\xi)\}\chi_1 u) > 0. \end{aligned}$$

(The last inequality follows from the fact that

$$\sigma^*(D_1)\sigma(D_1)(x,r\xi) \quad \text{and } \sigma(D_1)\sigma^*(D_1)(x,r\xi)$$

are self-adjoint matrix functions that are strictly positive for $r > 0$).

It follows that $\ker T = \ker \widetilde{T}$, where the operator

$$\widetilde{T} \simeq r^{-2m}\begin{pmatrix} \sigma^*(D_1)\sigma(D_1)(x,v(r)\xi) + \sigma_c^*(D_2)\sigma_c(D_2)\left(ir\frac{\partial}{\partial r}\right) & 0 \\ 0 & \sigma(D_1)\sigma^*(D_1)(x,v(r)\xi) + \sigma_c(D_2)\sigma_c^*(D_2)\left(ir\frac{\partial}{\partial r}\right) \end{pmatrix}, \tag{2.7}$$

differs from T in that r has been replaced by $v(r)$ wherever it is multiplied by ξ; here $v(r) = r$ for $r > \varepsilon$ and $v(r) > \varepsilon/2$. The operator

$$U = r^{2m}\left(v(r)\xi^2 - (r\partial/\partial r)^2 - \triangle_\Omega\right)^{-m}\widetilde{T}$$

has an operator-valued symbol satisfying the conditions of Definition 5.3 and hence is invertible (together with $\widetilde{T}$ and T) for sufficiently large $|\xi|$. It follows from the homogeneity properties of T that $T(\xi)$ is invertible for all $\xi \neq 0$, so that $\ker T = \ker \sigma_\Lambda(D_1 \# D_2)$. A similar computation (with reverse order of the factors) shows that $\operatorname{coker} \sigma_\Lambda(D_1 \# D_2)$ is also trivial.

Thus, the tensor product $D_1 \# D_2$ is an elliptic operator, and from the finiteness theorem (see [7]) we obtain the following result.

Theorem 2.1. *If the operator D_1 is elliptic and the operator D_2 is elliptic for the weight $\gamma = m/2$, where $m = \operatorname{ord} D_1 = \operatorname{ord} D_2$, then the tensor product*

$$D_1 \# D_2 : W^{s,\gamma}(\mathbb{W}) \longrightarrow W^{s-m,\gamma-m}(\mathbb{W}),$$

is a Fredholm edge-degenerate operator for the same weight.

We see that the tensor product construction produces an elliptic operator $D_1 \# D_2$ on a manifold with edges which, in contrast to the general theory (see [5]), does not require additional edge boundary and coboundary conditions.

3 Example 2. An operator on a manifold with edges admitting no well-posed boundary and coboundary conditions

In elliptic theory on manifolds with edges, there is an obstruction to the existence of well-posed edge boundary and coboundary conditions. In a sense, this obstruction is similar to the Atiyah–Bott obstruction [9] to the existence of well-posed classical boundary value problems. However, the question as to whether this obstruction is always trivial has been open yet. Using the tensor product technique, we give an example where the obstruction is nontrivial.

This time we again start from a closed manifold M_1 and a manifold M_2 with boundary. However, to this pair we assign a different manifold with edge ∂M_2, namely, the manifold

$$\mathbb{W}' = M_1 \times M_2 / \sim,$$

obtained from the Cartesian product of M_1 and M_2 by identifying all points of the boundary $M_1 \times \partial M_2$ lying in the same fiber of the projection $M_1 \times \partial M_2 \to \partial M_2$. The coordinates on M_1 will be denoted by ω, and the coordinates in a neighborhood of the boundary of M_2 by r, x.

Example 3.1. By way of an illustration, we take the circle $M_1 = \mathbb{S}^1$ and the two-dimensional disk $M_2 = \mathbb{D}^2$. Then the product $M = M_1 \times M_2$ is the solid torus, and the space $\overline{M}_2$ is homeomorphic to the two-dimensional sphere. The manifold with edge $\mathbb{W} = M_1 \times M_2$ is homeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^2$ of spheres. Furthermore, the space $\mathbb{W}$ itself can be obtained from the solid torus by identifying the meridians lying on the boundary. On the other hand, the manifold with edge $\mathbb{W}'$ considered in this section is obtained by identifying the parallels on the torus. One can readily see that the resulting space is homeomorphic to the sphere $\mathbb{S}^3$.

As before, let D_1 be an elliptic operator on M_1. Next, let D_2 be an elliptic operator on M_2 without degeneration and suppose that in a neighborhood of the boundary it has the form

$$D_2|_{U_{\partial M_2}} = \Gamma\left(\frac{\partial}{\partial r} + A\left(x, -i\frac{\partial}{\partial x}\right)\right),$$

where $A = A(x, -i\partial/\partial x)$ is an elliptic self-adjoint operator on the boundary with principal symbol $a(x,\xi)$ and Γ is an isomorphism of vector bundles. For brevity, we assume that Γ is the identity map.

Now we consider a neighborhood U of the boundary of $M_1 \times M_2$ and choose a diffeomorphism g_U of U on the product

$$[0,1) \times M_1 \times \partial M_2.$$

On the product $(0,1) \times M_1$, we consider the cone-degenerate (at $r=0$) elliptic operator

$$C = \frac{1}{r}\begin{pmatrix} -ir\frac{\partial}{\partial r} & D_1^* \\ D_1 & ir\frac{\partial}{\partial r} \end{pmatrix} : \begin{array}{c} C^\infty((0,1)\times M_1, E_1) \\ \oplus \\ C^\infty((0,1)\times M_1, F_1) \end{array} \longrightarrow \begin{array}{c} C^\infty((0,1)\times M_1, E_1) \\ \oplus \\ C^\infty((0,1)\times M_1, F_1). \end{array}$$

Its principal symbol is Hermitian and satisfies the interior ellipticity condition. Now consider the tensor product (1.4) of the self-adjoint operator A by C. The tensor product $A\#C$ is an operator on $[0,1) \times M_1 \times \partial M_2$ and has the form

$$A\#C = r^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} r\frac{\partial}{\partial r} + rA\left(x, -i\frac{\partial}{\partial x}\right) & D_1^*\left(\omega, -i\frac{\partial}{\partial \omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial \omega}\right) & r\frac{\partial}{\partial r} - rA\left(x, -i\frac{\partial}{\partial x}\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

It follows that for $r=1$ the resulting operator coincides (up to isomorphisms of vector bundles specified by the first and the third matrix factors in the product) with the operator

$$D_1\#D_2 = r^{-1}\begin{pmatrix} r\frac{\partial}{\partial r} + rA\left(x, -i\frac{\partial}{\partial x}\right) & D_1^*\left(\omega, -i\frac{\partial}{\partial \omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial \omega}\right) & r\frac{\partial}{\partial r} - rA\left(x, -i\frac{\partial}{\partial x}\right) \end{pmatrix}.$$

Thus on the wedge manifold we have the globally defined operator

$$D = (D_1\#D_2)\,\chi + g_U^*(A+iC)(1-\chi),$$

where $\chi \in C_0^\infty((0,1])$ is a cutoff function equal to 1 for $r > 9/10$. The resulting operator is edge degenerate. Let us verify the ellipticity.

A. The interior ellipticity for $r \neq 0$ is obvious, since the operator is the tensor product of elliptic operators. The interior ellipticity for $r = 0$ of the principal symbol also holds, since for $r = 0$ we have the symbol

$$\begin{pmatrix} ip + a(x,\xi) & \sigma^*(D_1)(\omega, q) \\ \sigma(D_1)(\omega, q) & ip - a(x,\xi) \end{pmatrix},$$

which is obviously the tensor product of invertible symbols.

B. Now let us find conditions under which the edge symbol of D is a Fredholm family.

Let $\sigma_\Lambda(D)(x,\xi)$ be the edge symbol of D. It is a family parametrized by $T^*\partial M_2 \setminus \{0\}$:

$$\sigma_\Lambda(D)(x,\xi) = r^{-1} \begin{pmatrix} r\frac{\partial}{\partial r} + ra(x,\xi) & D_1^*\left(\omega, -i\frac{\partial}{\partial \omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial \omega}\right) & r\frac{\partial}{\partial r} - ra(x,\xi) \end{pmatrix}.$$

The operator of the family act in the spaces

$$\sigma_\Lambda(D)(x,\xi) : \begin{matrix} \mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times M_1, E_1 \otimes E_{2,x}) \\ \oplus \\ \mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times M_1, E_1 \otimes F_{2,x}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-1,\gamma-1}(\mathbb{R}_+ \times M_1, E_1 \otimes E_{2,x}) \\ \oplus \\ \mathcal{K}^{s-1,\gamma-1}(\mathbb{R}_+ \times M_1, E_1 \otimes F_{2,x}). \end{matrix}$$

By $\sigma_c = \sigma_c(\sigma_\Lambda(D))$ we denote the conormal symbol of the edge symbol. It has the form

$$\sigma_c\left(x, \omega, -i\frac{\partial}{\partial \omega}\right) = \begin{pmatrix} -ip & D_1^* \\ D_1 & -ip \end{pmatrix} : \begin{matrix} H^s(M_1, E_1 \otimes E_{2,x}) \\ \oplus \\ H^s(M_1, E_1 \otimes F_{2,x}) \end{matrix} \longrightarrow \begin{matrix} H^{s-1}(M_1, E_1 \otimes E_{2,x}) \\ \oplus \\ H^{s-1}(M_1, E_1 \otimes E_{2,x}). \end{matrix} \tag{3.1}$$

Proposition 3.2. *For each $x \in \partial M_2$ and p on the real line, the family* (3.1) *has a single degeneracy point $p = 0$, and the corresponding root space is*

$$\ker D_1 \otimes E_{2,x} \oplus \ker D_1^* \otimes F_{2,x}.$$

The proof obviously follows from the representation of the conormal symbol in form of the sum

$$\sigma_c = -ipI + \begin{pmatrix} 0 & D_1^* \\ D_1 & 0 \end{pmatrix}.$$

□

Corollary 3.3. *There exists a sufficiently small $\varepsilon > 0$ such that for every point $(x,\xi) \in S^*(\partial M_1)$ the edge symbol $\sigma_\Lambda(D)(x,\xi)$*

$$\sigma_\Lambda(D) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times \partial M_2) \longrightarrow \mathcal{K}^{s-1,\gamma-1}(\mathbb{R}_+ \times \partial M_2)$$

is Fredholm for all nonzero weights in the interval $-\varepsilon < \gamma < \varepsilon$.

The proof follows from the ellipticity (interior and conormal) of the operator $\sigma_\Lambda(D)$ (see [5]).

We denote the index of the Fredholm family $\sigma_\Lambda(D)(x,\xi)$ parametrized by the cosphere bundle of the edge ∂M_2 by

$$\operatorname{ind}\sigma_\Lambda(D) \in K(S^*\partial M_2).$$

The definition of the families index can be found in [10]. It turns out that we can give an explicit expression for this index in the case of the tensor product.

Theorem 3.4. *The index of the family $\sigma_\Lambda(D)$ is equal to*

$$\operatorname{ind}\sigma_\Lambda(D) = \dim\ker D_1[L_+(\sigma(A))] + \dim\ker D_1^*[L_-(\sigma(A))] \in K(S^*\partial M_2), \qquad (3.2)$$

where by $L_+(\sigma(A)) \in \operatorname{Vect}(S^*\partial M_2)$ we denote the positive spectral subbundle of the invertible Hermitian symbol $\sigma(A)$ and $L_-(\sigma(A))$ is the complementary subbundle corresponding to negative eigenvalues.

The proof will be given in the next section, and now we just apply this result.

It was shown in [7] that an elliptic edge-degenerate operator admits a well-posed edge problem if and only if the index of the edge symbol is zero as an element of the quotient group

$$\operatorname{ind}\sigma_\Lambda(D) \in K(S^*\partial M_2)/\pi^*K(\partial M_2),$$

where $\pi : S^*\partial M_2 \to M_2$ is the natural projection.

Theorem 3.5. *In the quotient group $K(S^*\partial M_2)/\pi^*K(\partial M_2)$, the following formula holds for the index of the edge symbol:*

$$\operatorname{ind}\sigma_\Lambda(D) = \operatorname{ind}D_1[L_+(\sigma(A))]. \qquad (3.3)$$

Proof. Indeed, $L_+(A)$ and $L_-(A)$ are complementary subbundles in π^*E_2. Hence in the quotient group one has

$$[L_-(A)] = -[L_+(A)] \in K(S^*\partial M_2)/\pi^*K(\partial M_2).$$

Now the desired formula can be obtained from the expression (3.2) by a straightforward substitution. $\square$

Corollary 3.6. *If D_1 has a nonzero index and the operator A on the boundary corresponding to D_2 determines a nonzero element of the group $K(S^*\partial M_2)\otimes\mathbb{Q}/K(\partial M_2)\otimes\mathbb{Q}$, then the obstruction is nonzero for the corresponding edge-degenerate operator D.*

Let us give a specific example in which the element (3.3) is nonzero.

First, we consider the simplest local model.

Example 3.7. On the cylinder $M_2 = \mathbb{S}^1 \times [0,1)$, we consider the Cauchy–Riemann operator $D_2 = \partial/\partial r - i\partial/\partial\varphi$. Next, let $M_1 = \mathbb{S}^2$, and let D_1 be the Euler operator

$$D_1 = d + \delta : \Lambda^{ev}(\mathbb{S}^2) \longrightarrow \Lambda^{odd}(\mathbb{S}^2).$$

Then

$$\operatorname{ind} D_1 = \chi(\mathbb{S}^2) = 2,$$

$$L_+ (\sigma(A)) (\xi) = \begin{cases} \mathbb{C}, & \text{for } \xi > 0, \\ 0, & \text{for } \xi < 0. \end{cases}$$

This bundle is a generator of the group

$$K\left(S^*\mathbb{S}^1\right)/\pi^* K\left(\mathbb{S}^1\right) = \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}.$$

The corresponding operator in a neighborhood of the edge has the form

$$D\big|_{M_1 \times U_{\partial M_2}} = r^{-1} \begin{pmatrix} r\frac{\partial}{\partial r} - ir\frac{\partial}{\partial\varphi} & D_1^*\left(\omega, -i\frac{\partial}{\partial\omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial\omega}\right) & r\frac{\partial}{\partial r} + ir\frac{\partial}{\partial\varphi} \end{pmatrix}.$$

By applying Corollary 3.6, we find that this operator has no well-posed edge boundary and coboundary conditions.

One can readily show that the Cauchy–Riemann operator extends to be an elliptic operator on the two-dimensional disk $\mathbb{D}^2$, containing the cylinder M_2 as a collar neighborhood of the boundary. By reproducing the above argument, we arrive at an edge-degenerate operator on a closed manifold with edge.

4 Computation of the index of the edge symbol

Proof of Theorem 3.4. To simplify the computations, we assume that the principal symbol of A satisfies the equality $a^2(x,\xi) = |\xi|^2$.

1) For an arbitrary positive r, consider the family of conormal symbols

$$\sigma_\Lambda(D)(x, r\xi, p) = \begin{pmatrix} ip + ra(x,\xi) & D_1^*\left(\omega, -i\frac{\partial}{\partial\omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial\omega}\right) & ip - ra(x,\xi) \end{pmatrix}. \tag{4.1}$$

Obviously, the singular points of this family in the strip $|\operatorname{Im} p| < \varepsilon$ can be obtained by clockwise rotation by an angle of $\pi/2$ from the eigenvalues of the family of self-adjoint operators

$$\begin{pmatrix} ra(x,\xi) & D_1^*\left(\omega, -i\frac{\partial}{\partial\omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial\omega}\right) & -ra(x,\xi) \end{pmatrix} \tag{4.2}$$

with absolute value less than ε. These eigenvalues, as well as the corresponding eigenspaces, are described in the following lemma.

Lemma 4.1. *For $|\xi| = 1$, the small eigenvalues λ of the operator in* (4.2) *are equal to $\pm r$,, and for $\lambda = r$ the corresponding eigenspaces have the form*

$$\left(\begin{array}{c} \ker D_1 \otimes L_+ (a(x,\xi)) \\ \ker D_1^* \otimes L_- (a(x,\xi)) \end{array}\right),$$

while for $\lambda = -r$ we have

$$\left(\begin{array}{c} \ker D_1 \otimes L_- (a(x,\xi)) \\ \ker D_1^* \otimes L_+ (a(x,\xi)) \end{array}\right).$$

In particular, for $r \neq 0$ the zero eigenvalue is missing.

Proof. Consider the eigenvalue problem

$$\left(\begin{array}{cc} ra(x,\xi) & D_1^* \\ D_1 & -ra(x,\xi) \end{array}\right)\left(\begin{array}{c} u \\ v \end{array}\right) = \lambda \left(\begin{array}{c} u \\ v \end{array}\right). \tag{4.3}$$

By applying the latter operator once again, we obtain

$$\left(\begin{array}{cc} r^2 I + D_1^* D_1 & 0 \\ 0 & r^2 I + D_1 D_1^* \end{array}\right)\left(\begin{array}{c} u \\ v \end{array}\right) = \lambda^2 \left(\begin{array}{c} u \\ v \end{array}\right).$$

Hence we have

$$\lambda^2 = r^2 + \mu^2,$$

where μ^2 is a (nonnegative) eigenvalue of $D_1^* D_1$. Hence if λ is small, then $\mu = 0$ (it is constant) and r is small. Hence $\lambda = \pm r$, and for the eigenvectors we have

$$D_1 u = 0, \quad D_1^* v = 0.$$

Thus, system (4.3) can be reduced to the equations

$$r(a(x,\xi) \mp 1)u = 0, \quad r(a(x,\xi) \pm 1)v = 0.$$

For $r \neq 0$, we arrive at the desired inclusions

$$u \in \ker D_1 \otimes L_\pm (a(x,\xi)),$$

$$v \in \ker D_1^* \otimes L_\mp (a(x,\xi)).$$

One can easily verify that all pairs (u, v) satisfying these conditions are indeed eigenvectors of the operator (4.2).

The proof of the lemma is complete.

$\square$

Now let us split the half-line $r > 0$ into two intervals $(0, \varepsilon')$ and $(\varepsilon', +\infty)$ such that the family (4.1) is invertible on the weight line $\operatorname{Im} p = -\varepsilon$ on the closure of the first interval. It suffices to take a sufficiently small ε' and use the continuous dependence of the family (4.1) on the parameter r. On the remaining part $[\varepsilon', +\infty)$, the family is invertible on the weight line $\operatorname{Im} p = 0$ by Lemma 4.1.

By a small deformation of the family (4.1) of conormal symbols, we can ensure that this family is independent of r in a neighborhood of $r = \varepsilon'$. By $\widetilde{\sigma}_\Lambda(D)$ we denote the differential operator on the infinite cone K_{M_1} obtained for $r > \varepsilon'$ as a restriction of the operator

$$\widetilde{\sigma}_\Lambda(D) = r^{-m}\sigma_\Lambda(D)\left(x, \varepsilon'\xi, ir\frac{\partial}{\partial r}\right).$$

This operator is continuous in the weighted Sobolev spaces

$$\widetilde{\sigma}_\Lambda(D) : H^{s,-\varepsilon,0}(K_{M_1}) \longrightarrow H^{s-1,-\varepsilon-1,-1}(K_{M_1})$$

on the infinite cone and is Fredholm. To make the exposition more self-contained, we recall the definition of weighted Sobolev spaces on the cone (e.g., see [11]). Namely, the cone K_{M_1} can be treated as a manifold with two conical points $r = 0$ and $r = \infty$. (The radial variable in a neighborhood of the latter is $r' = 1/r$.) Then $H^{s,\gamma_1,\gamma_2}(K_{M_1})$ is the weighted Sobolev space of order s with weight γ_1 at $r = 0$ and weight $(-\gamma_2)$ at $r' = 0$.

Lemma 4.2. *One has*

$$\operatorname{ind} \sigma_\Lambda(D) = \operatorname{ind} \widetilde{\sigma}_\Lambda(D).$$

The proof will be given in Appendix II.

Now let us compute the index of the family $\widetilde{\sigma}_\Lambda(D)$ in weighted Sobolev spaces. Consider the following homotopy of the operators of the family:

$$\widetilde{\sigma}_\Lambda = r^{-m}\widetilde{\sigma}_\Lambda(D)\left(x, (r+\lambda)\xi, ir\frac{\partial}{\partial r}\right) \qquad \lambda \in [0, 1).$$

By construction, this is a homotopy of Fredholm operators, since the conormal symbol at infinity remains unchanged, and the conormal symbol at $r = 0$ remains invertible on the weight line $\operatorname{Im} p = -\varepsilon$. Moreover, at the end of the homotopy at $\lambda = 1$ the operator $\widetilde{\sigma}_1$ is equivalent to an operator with constant coefficients on the infinite cylinder with coordinate $t = \ln r$. The index of families of this type was computed in [14]. Moreover, the cokernel of the operator is trivial, and hence the index satisfies

$$\operatorname{ind} \widetilde{\sigma}_1 = [\ker \widetilde{\sigma}_1] \in K(S^*\partial M_2).$$

The latter operator has the form

$$\widetilde{\sigma}_1 = r^{-1}\begin{pmatrix} r\frac{\partial}{\partial r} + \varepsilon' a\,(x,\xi) & D_1^*\left(\omega, -i\frac{\partial}{\partial \omega}\right) \\ D_1\left(\omega, -i\frac{\partial}{\partial \omega}\right) & r\frac{\partial}{\partial r} - \varepsilon' a\,(x,\xi) \end{pmatrix}.$$

One can readily compute the kernel of this operator on K_{M_1}. Namely, the expansion in eigenfunctions of the self-adjoint operator

$$\begin{pmatrix} \varepsilon' a\left(x,\xi\right) & D_1^*\left(\omega,-i\frac{\partial}{\partial\omega}\right) \\ D_1\left(\omega,-i\frac{\partial}{\partial\omega}\right) & -\varepsilon' a\left(x,\xi\right) \end{pmatrix} \tag{4.4}$$

shows that the kernel of the operator $\widetilde{\sigma}_1$ in the weighted Sobolev spaces is isomorphic to the eigenspace of the operator (4.4) corresponding to small positive eigenvalues. By Lemma 4.1, this subspace is

$$L_+\left(\sigma\left(A\right)\right)\left(x,\xi\right)\otimes\ker D_1\oplus L_-\left(\sigma\left(A\right)\right)\left(x,\xi\right)\otimes\ker D_1^*, \tag{4.5}$$

which provides the global equality of vector bundles:

$$\ker\widetilde{\sigma}_1\simeq L_+\left(\sigma\left(A\right)\right)\otimes\ker D_1\oplus L_-\left(\sigma\left(A\right)\right)\otimes\ker D_1^*.$$

By Lemma 4.2, we arrive at (3.2).

The proof of Theorem 3.4 is complete.

□

5 Appendix I. Elliptic operators with parameter in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$

Let Ω be a smooth compact $(n-1)$-dimensional manifold. We shall consider operators depending on the parameter $\xi\in\mathbb{R}^l$ in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$ on the infinite cone K_Ω.

As symbols we take operator-valued functions (ranging in the space of operators in the Sobolev scale on Ω) of the variables $r\in\mathbb{R}_+$, $p\in\mathbb{R}$ (the conormal variable), and $\xi\in\mathbb{R}^l$ (additional parameters).

Definition 5.1. By $L^m\equiv L^m(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^l)$, where $m\le 0$, we denote the space of smooth functions

$$F:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^l\longrightarrow\bigcap\mathcal{L}\big(H^s(\Omega),H^{s-m}(\Omega)\big), \tag{5.1}$$

defined for all r, p, and sufficiently large $|\xi|$ and possessing the following properties:

1. For $r<R$, where R is an arbitrary positive number, $F(r,p,\xi)$ is an mth-order ψDO with parameters $(p,\xi)\in\mathbb{R}^{1+l}$ on Ω in the sense of Agranovich–Vishik [12] and smoothly depends on r as such a ψDO. Moreover, as $r\to 0$, all derivatives

$$\left(r\frac{\partial}{\partial r}\right)^k(r^mF(r,p,\xi)),\quad k=0,1,2\ldots,$$

are uniformly bounded as mth-order ψDO with parameters (and in particular as elements of the space $\bigcap\mathcal{L}\big(H^s(\Omega),H^{s-m}(\Omega)\big)$), and the family $r^mF(r,p,\xi)$ itself converges to some mth-order ψDO $F_0(p,\xi)$ with parameters (p,ξ).

2. For $r < R$, where R is an arbitrary positive number, the operator function

$$G(r,\eta,\xi) = F(r, r\eta, \xi), \quad \eta \in \mathbb{R}, \quad \xi \in \mathbb{R}^l, \tag{5.2}$$

satisfies the estimates

$$\left\| U^{k-m+\beta} \left[\frac{\partial^{\alpha+\beta}}{\partial r^\alpha \partial \eta^\beta} G(r,\eta,\xi) \right] U^{-k} \right\| \le C_{kj} r^{-\alpha}, \quad \alpha+\beta = j = 0,1,2,\dots, \quad k \in \mathbb{Z}, \tag{5.3}$$

where

$$U \equiv U(r,\eta,\xi) = \left(\xi^2 + \eta^2 - \frac{\triangle_\Omega}{r^2} \right)^{1/2}. \tag{5.4}$$

To each element $F \in L^m$, we assign a family of linear operators on $C_0^\infty(\mathbb{R}_+ \times \Omega)$ with parameters $\xi \in \mathbb{R}^l$ by setting

$$\widehat{F} = \chi_1 \circ F\left(\overset{2}{r}, ir\overset{1}{\frac{\partial}{\partial r}}, \xi \right) \circ \varphi_1 + \chi_2 \circ G\left(\overset{2}{r}, i\overset{1}{\frac{\partial}{\partial r}}, \xi \right) \circ \varphi_2, \tag{5.5}$$

where $\chi_1(r)$ and $\chi_2(r)$ are cutoff functions associated with φ_1 and φ_2. (Thus, $\chi_i \varphi_i = \varphi_i$, $\chi_1 = 0$ at infinity, and $\chi_2 = 0$ near zero.)

Theorem 5.2. *The following assertions hold.*

0) *For $F \in L^m$, the family $\widehat{F}$ is independent of the choice of the partition of unity and cutoff functions modulo families of the form $\widehat{G} + Q$, where $G \in L^{m-1}$ and Q satisfies the estimates*

$$\left\| Q(\xi) : \mathcal{K}^{s,0}(K_\Omega) \longrightarrow \mathcal{K}^{s',0}(K_\Omega) \right\| \le C_{ss'N} |\xi|^{-N}$$

for all s, s', N and is compact in all these pairs of spaces. (*The space of such operator families will be denoted by $\mathfrak{Q}$.*)

1) *If $F \in L^m$, then the operator*

$$\widehat{F}(\xi) : \mathcal{K}^{s,0} \longrightarrow \mathcal{K}^{s-m,0}$$

is continuous for all $\xi \neq 0$ uniformly in ξ for $|\xi| > \varepsilon > 0$.

2) *If $F \in L^{-1}$, then*

$$\left\| \widehat{F}(\xi) : \mathcal{K}^{s,0} \longrightarrow \mathcal{K}^{s,0} \right\| \le C_s |\xi|^{-1},$$

and for all s the operator $\widehat{F}(\xi)$ is compact in these spaces for $\xi \neq 0$.

3) *If* $F_j \in L^{m_j}$*,* $j = 1, 2$*, then*

$$\widehat{F}_1 \widehat{F}_2 = \widehat{F_1 F_2} \left(\mod \widehat{L}^{m_1+m_2-1} \right), \tag{5.6}$$

where by $\widehat{L}^{m_1+m_2-1}$ *we denote the space of operators of the form* $\widehat{F} + Q$*,* $F \in L^{m_1+m_2-1}$*,* $Q \in \mathfrak{Q}$*. If* $F \in L^m$*, then*

$$\widehat{F}^* = \widehat{\overline{A}} \left(\mod \widehat{L}^{m-1} \right), \tag{5.7}$$

where the star stands for the adjoint operator in $\mathcal{K}^{s,0}$ *for arbitrarily chosen* s*.*

The proof of items 0)–2) is based on item 3) and the obvious fact that for given N and negative m of sufficiently large absolute value one has the estimates

$$\left\| \widehat{F}(\xi) : \mathcal{K}^{s,0}(K_\Omega) \longrightarrow \mathcal{K}^{s+N,0}(K_\Omega) \right\| \leq C|\xi|^{-N}, \quad |s| < N, \tag{5.8}$$

and these operators are compact in the spaces indicated. The proof of item 3) is a standard exercise in noncommutative analysis (e.g., see [13]).

□

Definition 5.3. An operator-valued symbol $F \in L^0$ is said to be *elliptic* if it is invertible for sufficiently large $|\xi|$ and $F^{-1} \in L^0$.

Theorem 5.4. *If* $F \in L^0$ *is an elliptic symbol, then the family* $\widehat{F}(\xi)$ *consists of Fredholm operators of index zero invertible for sufficiently large* $|\xi|$*.*

The proof directly follows from 5.2.

□

6 Appendix II. The index of edge symbols in the spaces $\mathcal{K}^{s,\gamma}$ and $H^{s,\gamma}$

Theorem 6.1. *Let*

$$\widehat{D} = D\left(x, \overset{2}{r}\xi, ir\overset{1}{\frac{\partial}{\partial r}}\right) : \mathcal{K}^{s,0}(K_\Omega) \longrightarrow \mathcal{K}^{s,0}(K_\Omega), \quad (x,\xi) \in T^*X \setminus \{0\} \tag{6.1}$$

be a given elliptic edge symbol. Then there is an $R_0 > 0$ *such that*

$$\operatorname{ind} \widehat{D} = \operatorname{ind} \widehat{\mathcal{D}} \in K(S^*X); \tag{6.2}$$

here

$$\widehat{\mathcal{D}} : H^{s,0}(K_\Omega) \longrightarrow H^{s,0}(K_\Omega)$$

is the family of operators in weighted Sobolev spaces on the infinite cone given by the formula

$$\widehat{\mathcal{D}} = D\left(x, \varkappa(\overset{1}{r})\xi, ir\overset{2}{\frac{\partial}{\partial r}}\right), \tag{6.3}$$

where $\varkappa : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone nondecreasing function such that $r(0) = 0$ and $\varkappa(r) \equiv R_0$ for $r \geq R_0$.

Proof. We take an R_0 so large that the operator family $D(x, R_0\xi, p)$ be invertible for all $p \in \mathbb{R}$ and all ξ with $|\xi| \geq 1$. It follows from the relative index theorem for abstract elliptic families (see [15]) that

$$\operatorname{ind} \widehat{D} = \operatorname{ind} \widehat{\mathcal{D}} + \operatorname{ind} \widehat{D}_1, \tag{6.4}$$

where the operator D_1 is given by the expression

$$\widehat{D}_1 = D\left(x, (\overset{2}{r} + R_0)\xi, ir\overset{1}{\frac{\partial}{\partial r}}\right) : \mathcal{K}^{s,0}(K_\Omega) \longrightarrow \mathcal{K}^{s,0}(K_\Omega), \quad (x,\xi) \in S^*X. \tag{6.5}$$

The symbol

$$D_1 = D(x, (r + R_0)\xi, p) \tag{6.6}$$

of the resulting operator satisfies the conditions of Definition 5.3, so that for sufficiently large $\lambda > 1$ the family

$$\widehat{D}_{1\lambda} = D\left(x, (\overset{2}{r} + R_0)\lambda\xi, ir\overset{1}{\frac{\partial}{\partial r}}\right) : \mathcal{K}^{s,0}(K_\Omega) \longrightarrow \mathcal{K}^{s,0}(K_\Omega), \quad (x,\xi) \in S^*X \tag{6.7}$$

is invertible for all $(x, \xi) \in S^*X$. On the other hand, by the assumptions of the theorem, this family is elliptic for all $\lambda \geq 1$, so that

$$\operatorname{ind} \widehat{D}_{1\lambda} = \text{const} \equiv 0,$$

that is, $\operatorname{ind} \widehat{D}_1 = 0$. The proof of the theorem is complete.

$\square$

Proof of Lemma 4.2 We multiply the symbol $\sigma_\Lambda(D)$ on the left by the family

$$r^m (v(r)\xi^2 + (ir\partial/\partial r)^2 - \triangle_\Omega)^{-m/2},$$

whose index is zero. By applying Theorem 6.1 to the resulting family, we obtain

$$\operatorname{ind} r^m (v(r)\xi^2 + (ir\partial/\partial r)^2 - \triangle_\Omega)^{-m/2} \sigma_\Lambda(D) = \operatorname{ind} \varkappa(r)^m (v(r)\xi^2 + (ir\partial/\partial r)^2 - \triangle_\Omega)^{-m/2} \widetilde{\sigma}_\Lambda(D).$$

It remains to note that

$$\operatorname{ind} \varkappa(r)^m (v(r)\xi^2 + (ir\partial/\partial r)^2 - \triangle_\Omega)^{-m/2} = 0.$$

The proof is complete.

$\square$

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